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On Viscosity and Fluctuation-Dissipation in Exclusion Processes*

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We consider the asymmetric simple exclusion process. We review some results in dimension $d \geqslant 3$ concerning the fluctuation-dissipation theorem and we prove regularity of viscosity coefficients.

KEY WORDS: Exclusion process; fluctuation-dissipation theorem; regularity of viscosity coefficients.

1. INTRODUCTION

In this paper, we will review some of the results concerning the space-time fluctuations in equilibrium of the asymmetric simple exclusion process in dimension $d \geqslant 3$. These are examples of nonreversible models of interacting particle systems on $\mathbb{Z}^d$ with conservation of particles and a family of ergodic equilibrium distributions indexed by a single parameter, i.e., the density. The model is simple enough to admit a certain amount of explicit calculations.

Let $\hat{\xi}(x)$ be stationary random scalar field on $\mathbb{R}^d$ with a distribution ν that is invariant under translations by $x \in \mathbb{R}^d$. Under some reasonable conditions the distribution μ_ϵ of the field rescaled by

$$\xi_\epsilon(x) = \epsilon^{-\frac{d}{2}}[\hat{\xi}(\epsilon^{-1}x) - \alpha]$$

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* Dedicated to Gianni Jona Lasinio on his 70th birthday.

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where α is the mean $E^{\nu}[\hat{\xi}(x)]$, will converge, as $\epsilon \to 0$, to white noise μ with variance

$$E^{\mu}[\xi(x)\,\xi(y)] = \sigma^2 \delta(y-x)$$

If there is a Markovian evolution of the field $\hat{\xi}(\cdot)$ with ν as invariant measure then we have space time process $\hat{\xi}(x, t)$ with distribution $\hat{P}$ with marginals ν for any fixed time. If we now do a space-time rescaling of the form

$$\xi_\epsilon(x, t) = \epsilon^{-\frac{d}{2}}[\xi(\epsilon^{-1}x, \epsilon^{-2}t) - \alpha]$$

the scaled process $\xi_\epsilon(x, t)$ with distribution P_ϵ is expected to converge to an Ornstein–Uhlenbeck type fluctuation process $\xi(t, x)$ with distribution P, satisfying

$$d\xi(t, x) = (\mathscr{A}\xi)(t, x)\,dt + d\beta(x, t)$$

expressed in the weak form as a linear stochastic differential equation

$$d\langle G, \xi(t)\rangle = \langle \mathscr{A}G, \xi(t)\rangle\,dt + d\beta_G(t).$$

Usually $\mathscr{A}$ takes the form of an elliptic second order differential operator with constant coefficients

$$\mathscr{A}G = \sum_{i,j} a_{i,j} D_i D_j G$$

where $\{a_{i,j}\}$ is s symmetric positive definite matrix. The fluctuation-dissipation relation asserts that the family of Brownian motions $\beta_G(\cdot)$ that depend linearly on the test function G satisfy

$$E[[\beta_G(t)]^2] = \sigma^2 t \int_{\mathbb{R}^d} \langle a\,\nabla G, \nabla G\rangle\,dx$$

The proof usually follows the martingale methods. We can write a martingale decomposition (Itô's formula) of the form

$$d\langle G, \xi_\epsilon(t)\rangle = \langle G, \Psi_\epsilon(\xi_\epsilon(t))\rangle\,dt + dM_{\epsilon,G}(t)$$

We only have to check that

$$\lim_{\epsilon\to 0} \langle G, \Psi_\epsilon(\xi(\cdot))\rangle = \langle \mathscr{A}G, \xi\rangle$$

$$\lim_{\epsilon\to 0} E^{P_\epsilon}[[M_{\epsilon,G}(t)]^2] = \sigma^2 t \int_{\mathbb{R}^d} \langle a\,\nabla G, \nabla G\rangle$$

A central limit theorem for Martingales ensures that any limit of $M_{\epsilon,G}$ is a Brownian motion. Except for some technical issues this is the basic proof.

However there are models where something quite different happens. The term $\Psi_\epsilon(\xi)$ becomes big but its integral

$$\int_0^t \langle G, \Psi_\epsilon(\xi(s))\rangle\, ds \tag{1.1}$$

stays finite. Ψ_ϵ is represented as the sum $\phi_\epsilon^{(1)} + \phi_\epsilon^{(2)}$. The big piece $\phi_\epsilon^{(1)}$ is transformed, by a central limit theorem of sorts, into Brownian noise and combines with the noise $M_{\epsilon,G}(t)$ to provide the new $\beta_G(t)$. The remaining part $\phi_\epsilon^{(2)}$ stays finite and becomes

$$\int_0^t \langle \mathscr{A} G, \xi(s)\rangle\, ds$$

where $\mathscr{A}$ again is of the same form, i.e., a second order elliptic operator. This splitting has to done by a carefully constructed decomposition and a new formula provided for the coefficients $\{a_{i,j}\}$ that define $\mathscr{A}$.

The simple exclusion models studied here provide examples of both situations. The symmetric case is an easy text book case, while the asymmetric versions exhibit the more complex behavior alluded to earlier.

2. NOTATION AND RESULTS

Fix a probability distribution $p(\cdot)$ supported on a finite subset of $\mathbb{Z}_*^d = \mathbb{Z}^d \setminus \{0\}$ and denote by L the generator of the simple exclusion process on $\mathbb{Z}^d$ associated to $p(\cdot)$. L acts on local functions f on $\mathscr{X} = \{0,1\}^{\mathbb{Z}^d}$ as

$$(Lf)(\eta) = \sum_{x, y \in \mathbb{Z}^d} p(y-x)\, \eta(x)\{1-\eta(y)\}[f(\sigma^{x,y}\eta) - f(\eta)] \tag{2.1}$$

where $\sigma^{x,y}\eta$ stands for the configuration obtained from η by exchanging the occupation variables $\eta(x)$, $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z) & \text{if } \ z \neq x,\ y \\ \eta(x) & \text{if } \ z = y \\ \eta(y) & \text{if } \ z = x \end{cases}$$

The model describes a possibly infinite system of particles, with at most one particle per site on the lattice $\mathbb{Z}^d$ that evolves according to the following simple random motion. Each particle waits for a standard exponential time, i.e., for a Poisson event of rate 1 to occur, and then tries to jump to a new site that is picked at random. If the original site is x the probability of picking y is $p(y-x)$. If the new site is empty the jump is completed. If the site y is occupied the jump cannot be carried out and the particle is forced to wait for another exponential time. All the particles are performing this motion independently and concurrently. This is described formally by the generator (2.1).

For this model one can check that Bernoulli product measures with $\nu_\alpha[\eta(x)=1]=\alpha$ for $0\leqslant\alpha\leqslant 1$ are equilibrium distributions. This amounts to verifying that

$$\int_X (Lf)(\eta)\, \nu_\alpha(d\eta)=0$$

for every local function $f: X\to R$. It is not hard to check that these are actually extremal or ergodic for the evolution. If we fix $0<\alpha<1$, there is a stationary process η_t with values in X, which has ν_α as the distribution at any time t. If $p(\cdot)$ is symmetric about the origin, the generator (2.1) is self-adjoint with respect to ν_α and therefore the process η_t is reversible. In any case if $f(\eta)$ is a local function on X with mean 0 with respect to ν_α then the ergodic theorem asserts that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(\eta_t)\, dt=0$$

almost surely with respect to the process P_α, the stationary Markov process with evolution determined by (2.1) and invariant distribution ν_α. The existence of the conserved quantity results in the absence of a uniform rate in the ergodic theorem. In the reversible case this manifests itself in the absence of a spectral gap for the operaror L given by (2.1) in $L_2(\nu_\alpha)$. Let us start with the density field

$$\sum_{x\in\mathbb{Z}^d}\eta(x)\,\delta_x$$

Define its normalized and rescaled version

$$\xi_\epsilon(\cdot)=\epsilon^{\frac{d}{2}}\sum_x[\eta(x)-\alpha]\,\delta_{\epsilon x}$$

which is to be thought of as density fluctuation over a spatial scale of size ϵ^{-1} and express it in weak form as

$$g_\epsilon(\eta)=\langle G,\xi_\epsilon\rangle=\epsilon^{\frac{d}{2}}\sum_x G(\epsilon x)[\eta(x)-\alpha]$$

A computation reveals

$$(Lg_\epsilon)(\eta) = \epsilon^{\frac{d}{2}} \sum_x \sum_y p(y-x)\, \eta(x)(1-\eta(y))[G(\epsilon y) - G(\epsilon x)]$$

If we now assume that $p(\cdot)$ is symmetric,

$$\begin{aligned}(Lg_\epsilon)(\eta) &= \frac{\epsilon^{\frac{d}{2}}}{2} \sum_x \sum_y p(y-x)\, [\eta(x)(1-\eta(y)) - \eta(y)(1-\eta(x))][G(\epsilon y) - G(\epsilon x)] \\ &= \frac{\epsilon^{\frac{d}{2}}}{2} \sum_x \sum_y p(y-x)\, [\eta(x) - \eta(y)][G(\epsilon y) - G(\epsilon x)] \\ &= \epsilon^{\frac{d}{2}} \sum_x \sum_y p(y-x)\, \eta(x)\, [G(\epsilon y) - G(\epsilon x)] \\ &\simeq \epsilon^{\frac{d}{2}+2} \sum_x [\eta(x) - \alpha]\, \mathscr{A} G(\epsilon x)\end{aligned}$$

where $\mathscr{A}$ is the elliptic operator

$$\tfrac{1}{2} \sum_{i,j} a_{i,j} D_{i,j}$$

with

$$a_{i,j} = \sum_x p(x)\, x_i x_j$$

the covariance matrix of $p(\cdot)$. Therefore the density fluctuation

$$\xi_\epsilon = \epsilon^{\frac{d}{2}} \sum_x [\eta(x) - \alpha]\, \delta_{\epsilon x} \tag{2.2}$$

has an approximate dissipation that equals

$$\epsilon^2 \mathscr{A} \xi_\epsilon$$

Speeding up time by the factor ϵ^{-2} takes us to diffusive scaling, replacing L by $\epsilon^{-2} L$, and now the dissipation will be approximately $\mathscr{A}\xi_\epsilon$. There is noise that is built in, in the Poisson process generating the jumps, which scales in the diffusive scale to a Brownian noise, leading to an Ornstein–Uhlenbeck process for the density fluctuations in the diffusive scale. The equilibrium fluctuations are that of white noise

$$E[\langle \xi(t), G\rangle^2] = \alpha(1-\alpha) \int_{R^d} |G|^2\, dx$$

and the evolution of the fluctuations in the diffusive scale takes the form

$$d\langle \xi(t)\,, G\rangle = \langle G, \mathscr{A}\xi(t)\rangle\, dt + d\beta_G(t)$$

with

$$E[d\beta_G^2] = \alpha(1-\alpha)\left[\int_{R^d} \sum_{i,j} a_{i,j}(D_i G)(D_j G)\, dx\right] dt$$

If we drop the assumption of symmetry on $p(\cdot)$, the situation is much more complex. Let us still suppose that

$$\sum_z z p(z) = 0$$

We cannot proceed beyond

$$(Lg_\epsilon)(\eta) = \epsilon^{\frac{d}{2}} \sum_x \sum_y p(y-x)\, \eta(x)(1-\eta(y))[G(\epsilon y) - G(\epsilon x)]$$

which we can rewrite as

$$(Lg_\epsilon)(\eta) \simeq \epsilon^{\frac{d}{2}+1} \sum_x \sum_y p(y-x)[\eta(x)(1-\eta(y)) - \alpha(1-\alpha)]\langle y-x, \nabla G(\epsilon x)\rangle$$

The rescaling, which should still be by a factor of ϵ^{-2} leads us to a term

$$\begin{aligned}\Psi_\epsilon(\xi) &= \epsilon^{\frac{d}{2}-2} \sum_x \sum_y p(y-x)\, \eta(x)(1-\eta(y))[G(\epsilon y) - G(\epsilon x)]\\ &\simeq \epsilon^{\frac{d}{2}-1} \sum_x \langle \nabla G(\epsilon x), \tau_x W(\eta)\rangle\end{aligned}$$

with $W(\eta) = \{W_i(\eta)\}$ given by

$$W_i(\eta) = \eta(0) \sum_z p(z)\, z_i(1-\eta(z))$$

which is of magnitude ϵ^{-1}. In the reversible case $W_i(\eta)$ is of a special form

$$W_i(\eta) = \sum_z c_i(z)[\eta(z) - \eta(0)]$$

allowing for another summation by parts, getting rid of the troublesome ϵ^{-1} factor and yielding, as earlier,

$$\Psi_\epsilon(\xi) \simeq \langle \mathscr{A} G, \xi_\epsilon\rangle$$

for some second order elliptic differential operator $\mathscr{A}$ that can be calculated. More generally if we have a *gradient system*, i.e., if W_i takes the form

$$W_i(\eta) = \sum_z c_i(z)[\tau_z h(\eta)) - h(\eta)]$$

for some local h with zero mean, then

$$\Psi_\epsilon(\xi) = \epsilon^{\frac{d}{2}} \sum_x (\mathscr{A}G)(\epsilon x)\, \tau_x h(\eta)$$

In such a case one can replace Ψ_ϵ by

$$\hat{\Psi}_\epsilon = c\epsilon^{\frac{d}{2}} \sum_x (\mathscr{A}G)(\epsilon x)[\eta(x) - \alpha]$$

where the constant c is calculated as

$$c = \frac{d}{d\alpha} E^{\nu_\alpha}[h(\eta)]$$

In general if $W(\eta)$ is not of gradient type then we show that $W(\eta)$ can be replaced by $\sum_z c(W, z)[\eta(z) - \eta(0)]$ for a suitable choice of $c(W, z)$.

Finally when $\sum_z zp(z) = m \neq 0$, it turns out that if we speed up time by ϵ^{-1}, then there is a limit for the fluctuations and it is

$$d\xi(x, t) + (1 - 2\alpha)\langle m, \nabla\xi(x, t)\rangle = 0$$

In that scaling the Poisson noise become ineffective and disappears leaving just a translation. If we denote $\gamma = m(1 - 2\alpha)$, then we can center the translations and consider

$$\xi_\epsilon(x - \gamma\epsilon^{-1}t, \epsilon^{-2}t)$$

This will now have a scaling limit and the analysis is very similar to the one for the mean 0 asymmetric case.

In order to carry out all the steps we need to understand the behavior of

$$\int_0^T \sum_{z \in V} \tau_z f(\eta_t)\, dt$$

over large volumes and large times. We can take V to be the box $\{x: |x_i| \leqslant \epsilon^{-1}; 1 \leqslant i \leqslant d\}$. We can take T to be either ϵ^{-1} or ϵ^{-2}. With the choice of $T = \epsilon^{-1}$, we can show that

$$\epsilon^{\frac{d}{2}} \int_0^T \left[\sum_{z \in V} \tau_z(f(\eta_t) - c(f, \alpha)\, \eta(0)) \right] dt$$

is negligible if we choose $c(f, \alpha)$ to be

$$c(f, \alpha) = \frac{d}{d\alpha} E^{\nu_\alpha}[f(\eta)]$$

Now we turn to $T = \epsilon^{-2}$. The main result is Theorem 5.3, known as the fluctuation-dissipation theorem. There are two distinct types of fluctuations. If u is a local function, then fluctuations of the form

$$\int_0^t f_\epsilon(\eta_{s\epsilon^{-2}})\, ds$$

where

$$f_\epsilon(\eta) = \sum_x G(\epsilon x)(\tau_x Lu)(\eta)$$

satisfy a central limit theorem and converge to a Brownian Motion when scaled by $\epsilon^{1+\frac{d}{2}}$.

On the other hand if $v = \sum_z b(z)[\eta(z) - \eta(0)]$ for some b supported on a finite subset of $\mathbb{Z}^d_*$, then

$$\begin{aligned} g_\epsilon &= \sum_x G(\epsilon x)(\tau_x v)(\eta) = \sum_{x, z} G(\epsilon x)\, b(z)[\eta(x+z) - \eta(x)] \\ &= \sum_x G(\epsilon x)\, \tau_x h(\eta) \end{aligned}$$

with $h(\eta) = \sum_z b(z)[\eta(z) - \eta(0)]$ and as we saw before, a summation by parts helps. Consequently fluctuations of the form

$$\int_0^t g_\epsilon(\eta_s \epsilon^{-2})\, ds$$

when normalized again by $\epsilon^{1+\frac{d}{2}}$ correspond to a time average of density fluctuations.

The analysis of *non-gradient systems* depends on our ability to write any W that satisfies some conditions as

$$W = Lu + h$$

While u and h may not be local functions, what we do show is that it is possible to approximate W well enough by $Lu+h$ with some local u and h.

The fluctuation dissipation theorem asserts that if $w \in \bigoplus_{n \geqslant 2} \mathscr{G}_n$ (see the end of the section for a definition) is a local function then there are coefficients $b(z, w)$ such that the fluctuation of $w - \sum_z b(z, w)[\eta(z) - \eta(0)]$ is well approximated by fluctuations of the first type. The constants $b(z, w)$ are defined for each α. We prove in Section 7 that these are C^∞ functions of α.

The study of fluctuations therefore depends on controlling space-time correlations of the form

$$E^{\nu_\alpha}\left[\left|\int_0^T \sum_{z \in V} \tau_z f(\eta_t)\, dt\right|^2\right]$$

for large times T and large volumes V.

The natural way to control such objects is to invert the generator of the process L, i.e., to solve the equation

$$Lu = \sum_{z \in V} \tau_z f \tag{2.3}$$

perhaps with $u = \sum_{z \in V} \tau_z v$ for some v.

Our goal is to show that we can approximate, in a proper weak sense, the solution of equation (2.3) by such functions. Our interest is to perform such approximation in a diffusive space-time scaling limit (i.e., $T \sim |V|^{2/d}$).

We will characterize the functions f such that the corresponding asymptotic space-time variance is finite and then we will prove that this variance is a smooth function of the density α.

There is a natural orthonormal basis for $L_2(\nu_\alpha)$ indexed by finite subsets A of Z^d that commutes with τ_z. Moreover there is a corresponding decomposition of $L_2(\nu_\alpha)$ as $\bigoplus_{n \geqslant 0} \mathscr{G}_n$ as the direct sum of orthogonal subspaces according to the cardinality n of A. Since, in this context, one cannot distinguish between f and $\tau_z f$, A and $\tau_z A$ can be identified. A reasonable cross section, one with a multiplicity of n for sets of cardinality n can be found for the equivalence classes. Another important point is that that in the symmetric case L leaves each $\mathscr{G}_n$ invariant so that the inversion needs to be done only on each $\mathscr{G}_n$, which is no harder than the analysis of random walk of $n-1$ particles on Z^d. If $d \geqslant 3$, the asymmetric case can be treated as a perturbation of the symmetric one. Although this is a somewhat singular perturbation the transience of the random walk in $d \geqslant 3$ and the fact that the perturbation only causes $\mathcal{G}_n$ to be mapped into $\mathcal{G}_{n-1} \oplus \mathcal{G}_n \oplus \mathcal{G}_{n+1}$ help the analysis. The perturbation is controlled with the use of weighted norms with weights growing polynomially in the degree n of $\mathcal{G}_n$.

Duality for the symmetric simple exclusion was first observed by Spitzer and was broadly used in this context (cf. ref. 12). The use of a dual base in order to study fluctuations in the asymmetric simple exclusion in dimension $d \geqslant 3$ was introduced in ref. 8 (where the fluctuation-dissipation theorem was first proved) and in ref. 13. The fluctuation dissipation theorem was then applied in order to study the diffusive incompressible limit (cf. ref. 3), the first order corrections to the hydrodynamic limit (cf. ref. 6) and the equilibrium fluctuations for the density field.[2]

Always with the use of this dual base, in ref. 11 we prove the regularity of the selfdiffusion coefficient (as function of the density) for the *symmetric* simple exclusion (reversible). Using this approach in ref. 1 is proved the regularity of the bulk diffusion coefficient for a non-gradient speed change exclusion that has ν_α as equilibrium (reversible) measures.

3. DUALITY

We examine in this section the action of the symmetric part L^s of the generator on the space of local functions endowed with a particular scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. Fix once for all a density α in (0, 1). All expectations in this section are taken with respect to ν_α and we omit all sub-indices.

3.1. The Dual Space

For each $n \geqslant 0$, denote by $\mathcal{E}_n$ the subsets of $\mathbb{Z}^d$ with n points and let $\mathcal{E} = \bigcup_{n \geqslant 0} \mathcal{E}_n$ be the class of finite subsets of $\mathbb{Z}^d$. For each A in $\mathcal{E}$, let Ψ_A be the local function

$$\Psi_A = \prod_{x \in A} \frac{\eta(x) - \alpha}{\sqrt{\chi(\alpha)}}$$

where $\chi(\alpha) = \alpha(1-\alpha)$. By convention, $\Psi_\phi = 1$. It is easy to check that $\{\Psi_A, A \in \mathcal{E}\}$ is an orthonormal basis of $L^2(\nu_\alpha)$. For each $n \geqslant 0$, denote by $\mathcal{G}_n$ the subspace of $L^2(\nu_\alpha)$ generated by $\{\Psi_A, A \in \mathcal{E}_n\}$, so that $L^2(\nu_\alpha) = \bigoplus_{n \geqslant 0} \mathcal{G}_n$. Functions in $\mathcal{G}_n$ are said to have degree n.

Consider a local function f. Since $\{\Psi_A: A \in \mathscr{E}\}$ is a basis of $L^2(\nu_\alpha)$, we may write

$$f = \sum_{n \geqslant 0} \sum_{A \in \mathscr{E}_n} \mathfrak{f}(A) \Psi_A$$

Note that the coefficients $\mathfrak{f}(A)$ depend not only on f but also on the density α: $\mathfrak{f}(A) = \mathfrak{f}(A, \alpha)$. Since f is a local function, $\mathfrak{f}: \mathscr{E} \to \mathbb{R}$ is a function of finite support.

For local functions u, v, define the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ by

$$\langle\!\langle u, v \rangle\!\rangle = \sum_{x \in \mathbb{Z}^d} \{\langle \tau_x u, v \rangle - \langle u \rangle \langle v \rangle\} \tag{3.1}$$

where $\{\tau_x, x \in \mathbb{Z}^d\}$ is the group of translations. That this is in fact an inner product can be seen by the relation

$$\langle\!\langle u, v \rangle\!\rangle = \lim_{V \uparrow \mathbb{Z}^d} \frac{1}{|V|} \left\langle \sum_{x \in V} \tau_x(u - \langle u \rangle), \sum_{x \in V} \tau_x(v - \langle v \rangle) \right\rangle.$$

Since $\langle\!\langle u - \tau_x u, v \rangle\!\rangle = 0$ for all x in $\mathbb{Z}^d$, this scalar product is only semidefinite. Denote by $L^2_{\langle\!\langle \cdot, \cdot \rangle\!\rangle}(\nu_\alpha)$ the Hilbert space generated by the local functions and the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. The scalar product of two local functions u, v can be written in terms of the Fourier coefficients of u, v through a simple formula. To this end, fix two local functions u, v and write them in the basis $\{\Psi_A, A \in \mathscr{E}\}$:

$$u = \sum_{A \in \mathscr{E}} \mathfrak{u}(A) \Psi_A, \qquad v = \sum_{A \in \mathscr{E}} \mathfrak{v}(A) \Psi_A$$

An elementary computation shows that

$$\langle\!\langle u, v \rangle\!\rangle = \sum_{x \in \mathbb{Z}^d} \sum_{n \geqslant 1} \sum_{A \in \mathscr{E}_n} \mathfrak{u}(A) \mathfrak{v}(A + x)$$

In this formula, $B + z$ is the set $\{x + z; x \in B\}$. The summation starts from $n = 1$ due to the centering by the mean in the definition of the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

We say that two finite subsets A, B of $\mathbb{Z}^d$ are equivalent if one is the translation of the other. This equivalence relation is denoted by $\sim$ so that $A \sim B$ if $A = B + x$ for some x in $\mathbb{Z}^d$. Let $\tilde{\mathscr{E}}_n$ be the quotient of $\mathscr{E}_n$ with respect to this equivalence relation: $\tilde{\mathscr{E}}_n = \mathscr{E}_n/_\sim$, $\tilde{\mathscr{E}} = \mathscr{E}/_\sim$. If, for some n, $\mathfrak{f}: \mathscr{E}_n \to \mathbb{R}$ is a summable function,

$$\sum_{A \in \mathscr{E}_n} \mathfrak{f}(A) = \sum_{\tilde{A} \in \tilde{\mathscr{E}}_n} \tilde{\mathfrak{f}}(\tilde{A})$$

where, for any equivalence class $\tilde{A}$ and a summable function $\mathfrak{f}: \mathscr{E} \to \mathbb{R}$,

$$\tilde{\mathfrak{f}}(\tilde{A}) = \sum_{z \in \mathbb{Z}^d} \mathfrak{f}(A+z) \tag{3.2}$$

A being any representative from $\tilde{A}$.

In particular, for two local functions u, v,

$$\begin{aligned}\langle\!\langle u, v \rangle\!\rangle &= \sum_{x, z \in \mathbb{Z}^d} \sum_{n \geqslant 1} \sum_{\tilde{A} \in \tilde{\mathscr{E}}_n} \mathfrak{u}(A+z)\, \mathfrak{v}(A+x+z) \\ &= \sum_{n \geqslant 1} \sum_{\tilde{A} \in \tilde{\mathscr{E}}_n} \tilde{\mathfrak{u}}(\tilde{A})\, \tilde{\mathfrak{v}}(\tilde{A})\end{aligned}$$

We say that a function $\mathfrak{f}: \mathscr{E} \to \mathbb{R}$ is translation invariant if $\mathfrak{f}(A+x) = \mathfrak{f}(A)$ for all sets A in $\mathscr{E}$ and all sites x of $\mathbb{Z}^d$. Of course, functions $\tilde{\mathfrak{f}}$ on $\tilde{\mathscr{E}}$ are the same as translation invariant functions on $\mathscr{E}$. Fix a subset A of $\mathbb{Z}^d$ with n points. There are n sets in the class of equivalence of A that contain the origin.

Therefore, summing a translation invariant function $\mathfrak{f}$ over all equivalence classes $\tilde{A}$ in $\tilde{\mathscr{E}}_n$ is the same as summing $\mathfrak{f}$ over all sets B in $\mathscr{E}_n$ which contain the origin and dividing by n:

$$\sum_{\tilde{A} \in \tilde{\mathscr{E}}_n} \mathfrak{f}(\tilde{A}) = \frac{1}{n} \sum_{\substack{A \in \mathscr{E}_n \\ A \ni 0}} \mathfrak{f}(A)$$

provided $\mathfrak{f}(A) = \mathfrak{f}(A+x)$ for all A, x. Let $\mathscr{E}_*$ be the class of all finite subsets of $\mathbb{Z}^d_* = \mathbb{Z}^d \setminus \{0\}$ and let $\mathscr{E}_{*,n}$ be the class of all subsets of $\mathbb{Z}^d_*$ with n points. Then, we may write

$$\begin{aligned}\langle\!\langle u, v \rangle\!\rangle &= \sum_{n \geqslant 1} \frac{1}{n} \sum_{\substack{A \in \mathscr{E}_n \\ A \ni 0}} \tilde{\mathfrak{u}}(A)\, \tilde{\mathfrak{v}}(A) \\ &= \sum_{n \geqslant 0} \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \tilde{\mathfrak{u}}(A \cup \{0\})\, \tilde{\mathfrak{v}}(A \cup \{0\})\end{aligned}$$

In conclusion, if for a finitely supported function $\mathfrak{f}: \mathscr{E} \to \mathbb{R}$, we define $\mathfrak{T}\mathfrak{f}: \mathscr{E}_* \to \mathbb{R}$ by

$$(\mathfrak{T}\mathfrak{f})(A) = \tilde{\mathfrak{f}}(A \cup \{0\}) = \sum_{z \in \mathbb{Z}^d} \mathfrak{f}([A \cup \{0\}] + z)$$

we have that

$$\langle\!\langle u, v \rangle\!\rangle = \sum_{n \geqslant 0} \frac{1}{n+1} \sum_{A \in \mathscr{E}_{*,n}} \mathfrak{T}\mathfrak{u}(A)\, \mathfrak{T}\mathfrak{v}(A) \tag{3.3}$$

For $n \geqslant 0$, denote by π_n the projection that corresponds to $\mathscr{E}_{*,n}$ i.e., $(\pi_n \mathfrak{f})(A) = \mathfrak{f}(A)\, \mathbf{1}\{A \in \mathscr{E}_{*,n}\}$ and denote by $\langle \cdot, \cdot \rangle$ the usual scalar product on each set $\mathscr{E}_{*,n}$: for $\mathfrak{f}$, $\mathfrak{g}$: $\mathscr{E}_{*,n} \to \mathbb{R}$,

$$\langle \mathfrak{f}, \mathfrak{g} \rangle = \sum_{A \in \mathscr{E}_{*,n}} \mathfrak{f}(A)\, \mathfrak{g}(A)$$

In view of formula (3.3), it is natural to introduce, for an integer $k \geqslant -1$, the Hilbert spaces $L^2(\mathscr{E}_*, k)$ generated by finite supported functions $\mathfrak{f}$: $\mathscr{E}_* \to \mathbb{R}$ and the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{0,k}$ defined by

$$\langle\!\langle \mathfrak{f}, \mathfrak{g} \rangle\!\rangle_{0,k} = \sum_{n \geqslant 0} (n+1)^k \langle \pi_n \mathfrak{f}, \pi_n \mathfrak{g} \rangle$$

With this notation, for local functions f, g in $\bigoplus_{n \geqslant 1} \mathscr{G}_n$,

$$\langle\!\langle f, g \rangle\!\rangle = \langle\!\langle \mathfrak{T}\mathfrak{f}, \mathfrak{T}\mathfrak{g} \rangle\!\rangle_{0,-1}$$

To summarize some observations on the transformation $\mathfrak{T}$, we need some notation. For a subset A of $\mathbb{Z}_*^d$ and x, y, z in $\mathbb{Z}_*^d$, denote by $A_{x,y}$ the set defined by

$$A_{x,y} = \begin{cases} (A \setminus \{x\}) \cup \{y\} & \text{if } \; x \in A,\ y \notin A \\ (A \setminus \{y\}) \cup \{x\} & \text{if } \; y \in A,\ x \notin A \\ A & \text{otherwise} \end{cases} \tag{3.4}$$

and denote by $S_z A$ the set defined by

$$S_z A = \begin{cases} A - z & \text{if } \; z \notin A \\ (A - z)_{0,-z} & \text{if } \; z \in A \end{cases} \tag{3.5}$$

Therefore, to obtain $S_z A$ from A in the case where z belongs to A, we first translate A by $-z$, getting a new set which contains the origin, and we then remove the origin and add site $-z$.

Remark 3.1.

(a) The restriction of $\mathfrak{f}$ to $\mathscr{E}_1$ is irrelevant for the definition of $\mathfrak{T}\mathfrak{f}(A)$ if A is nonempty.

(b) Not every function $\mathfrak{f}_*: \mathscr{E}_* \to \mathbb{R}$ is the image by $\mathfrak{T}$ of some function $\mathfrak{f}: \mathscr{E} \to \mathbb{R}$ since

$$(\mathfrak{T}\mathfrak{f})(A) = (\mathfrak{T}\mathfrak{f})(S_z A) \tag{3.6}$$

for all z in A.

(c) Let $\mathfrak{f}_*: \mathscr{E}_* \to \mathbb{R}$ be a finitely supported function satisfying (3.6): $\mathfrak{f}_*(A) = \mathfrak{f}_*(S_z A)$ for all z in A. Define $\mathfrak{f}: \mathscr{E} \to \mathbb{R}$ by

$$\mathfrak{f}(B) = \begin{cases} |B|^{-1}\, \mathfrak{f}_*(B \setminus \{0\}) & \text{if} \quad B \ni 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

An elementary computations shows that $\mathfrak{T}\mathfrak{f} = \mathfrak{f}_*$. With this choice, which is natural but not unique, $\mathfrak{f}(0) = \mathfrak{f}_*(\phi)$.

(d) $\mathfrak{T}$ maps $\mathscr{E}_n$ into $\mathscr{E}_{*,n-1}$ lowering the degree of a function by one. Thus the translations in the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ effectively reduce the degree by one while replacing the space $\mathbb{Z}^d$ by $\mathbb{Z}^d_*$.

(e) Formula (3.3) shows also that a local function f is in the kernel of the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ if and only if $\mathfrak{T}\mathfrak{f}$ vanishes, i.e., if and only if

$$\sum_{x \in \mathbb{Z}^d} \mathfrak{f}(A + x) = 0$$

for all finite subsets A such that $|A| \geqslant 1$. Examples of such functions are the constants or the difference of the translations of a local function: $\tau_y f - \tau_x f$.

To keep notation simple, most of the time, for a function $\mathfrak{f}: \mathscr{E} \to \mathbb{R}$, we denote $\mathfrak{T}\mathfrak{f}$ by $\bar{\mathfrak{f}}$. Real functions on $\mathscr{E}$ or on $\mathscr{E}_*$ are indistinctively denoted by the symbols $\mathfrak{f}$, $\mathfrak{g}$.

3.2. Some Hilbert Spaces

We investigate in this subsection the action of the symmetric part of the generator L on the basis $\{\Psi_A, A \in \mathscr{E}_*\}$.

Fix a function u of degree $n \geqslant 1$ and denote by $\mathfrak{u}$ its Fourier coefficients. A straightforward computation shows that

$$L^s u = \sum_{A \in \mathscr{E}_n} (\mathscr{L}_s \mathfrak{u})(A)\, \Psi_A \tag{3.8}$$

where $\mathscr{L}_s$ is the generator of finite symmetric random walks evolving with exclusion on $\mathbb{Z}^d$:

$$(\mathscr{L}_s \mathfrak{u})(A) = (1/2) \sum_{x,\, y \in \mathbb{Z}^d} s(y-x)[\mathfrak{u}(A_{x,\,y}) - \mathfrak{u}(A)] \tag{3.9}$$

and $A_{x,\,y}$ is the set defined by (3.4). Furthermore, an elementary computation, based on the fact that

$$\sum_{z \in \mathbb{Z}^d} \mathfrak{f}([B \cup \{y\}] + z) = \mathfrak{T}\mathfrak{f}(S_y B)$$

for all subsets B of $\mathbb{Z}_*^d$, sites y not in B and finitely supported functions $\mathfrak{f}\colon \mathscr{E} \to \mathbb{R}$, shows that for every set B in $\mathscr{E}_*$

$$\mathfrak{T}\mathscr{L}_s \mathfrak{u}\ (B) = \mathfrak{L}_s \mathfrak{T}\mathfrak{u}\ (B) \tag{3.10}$$

where

$$(\mathfrak{L}_s \bar{\mathfrak{u}})\ (B) = (1/2) \sum_{x,\, y \in \mathbb{Z}_*^d} s(y-x)[\bar{\mathfrak{u}}(B_{x,\,y}) - \bar{\mathfrak{u}}(B)]$$
$$+ \sum_{y \notin B} s(y)[\bar{\mathfrak{u}}(S_y B) - \bar{\mathfrak{u}}(B)]$$

This computation should be understood as follows. We introduced an equivalence relation in $\mathscr{E}$ when we decided not to distinguish between a set and its translations. This is the same as assuming that all sets contain the origin. If n particles evolve as exclusion random walks on $\mathbb{Z}^d$, one of them fixed to be at the origin, two things may happen. Either one of the particles which is not at the origin jumps or the particle we assumed to be at the origin jumps. In the first case, this is just a jump on $\mathbb{Z}_*^d$ and is taken care by the first piece of the generator $\mathfrak{L}_s$. In the second case, however, since we are imposing the origin to be always occupied, we need to translate back the configuration to the origin. This part corresponds to the second piece of the generator $\mathfrak{L}_s$.

We are now in a position to define the Hilbert space induced by the local functions $\mathscr{C}$, the symmetric part of the generator L and the scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. For two local functions u, v, let

$$\langle\!\langle u, v \rangle\!\rangle_1 = \langle\!\langle u, (-L^s)\, v \rangle\!\rangle$$

and let $H_1 = H_1(\mathscr{C}, L^s, \langle\!\langle \cdot,\cdot \rangle\!\rangle)$ be the Hilbert space generated by local functions f and the inner product $\langle\!\langle \cdot,\cdot \rangle\!\rangle_1$. By (3.8), (3.3), and (3.10) the previous scalar product is equal to

$$-\sum_{n\geqslant 0} \frac{1}{n+1} \sum_{A\in\mathscr{E}_{*,n}} \bar{\mathfrak{u}}(A)\, \overline{\mathscr{L}_s \mathfrak{v}}(A) = -\sum_{n\geqslant 0} \frac{1}{n+1} \sum_{A\in\mathscr{E}_{*,n}} \bar{\mathfrak{u}}(A)\, (\mathfrak{L}_s \bar{\mathfrak{v}})(A)$$
$$= \sum_{n\geqslant 0} \frac{1}{n+1} \langle \pi_n \bar{\mathfrak{u}}\,,\, (-\mathfrak{L}_s)\, \pi_n \bar{\mathfrak{v}} \rangle$$

because $\mathfrak{L}_s$ keeps the degree of the functions mapping $\mathscr{E}_{*,n}$ in $\mathscr{E}_{*,n}$. This formula leads to the following definitions. For each $n\geqslant 0$, denote by $\langle \cdot,\cdot \rangle_1$ the scalar product on $\mathscr{E}_{*,n}$ defined by

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_1 = \langle \mathfrak{f}, (-\mathfrak{L}_s)\, \mathfrak{g} \rangle$$

and denote by $\mathfrak{H}_1(\mathscr{E}_{*,n})$ the Hilbert space on $\mathscr{E}_{*,n}$ induced by the finitely supported functions and the scalar product $\langle \cdot,\cdot \rangle_1$. The associated norm is denoted by $\|\mathfrak{f}\|_1^2 = \langle \mathfrak{f}, (-\mathfrak{L}_s)\, \mathfrak{f} \rangle$. Furthermore, for an integer $k \geqslant -1$, denote by $\mathfrak{H}_{1,k} = \mathfrak{H}_1(\mathscr{E}_*, \mathfrak{L}_s, k)$ the Hilbert space induced by the finite supported functions $\mathfrak{f}, \mathfrak{g}\colon \mathscr{E}_* \to \mathbb{R}$ and scalar product

$$\langle\!\langle \mathfrak{f}, \mathfrak{g} \rangle\!\rangle_{1,k} = \langle\!\langle \mathfrak{f}, (-\mathfrak{L}_s)\, \mathfrak{g} \rangle\!\rangle_{0,k} = \sum_{n\geqslant 0} (n+1)^k \langle \pi_n \mathfrak{f}, (-\mathfrak{L}_s)\, \pi_n \mathfrak{g} \rangle$$

The associated norm is denoted by $\|\cdot\|_{1,k}$ so that $\|\mathfrak{f}\|_{1,k}^2 = \langle\!\langle \mathfrak{f}, \mathfrak{f} \rangle\!\rangle_{1,k}$

Three observations are in order. First of all, in the definition of the scalar product $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{1,k}$, because $\mathfrak{L}_s \mathfrak{f}(\phi) = 0$ for all $\mathfrak{f}$, it is irrelevant if the summation starts from $n=0$ or from $n=1$. On the other hand, it follows from the previous relation that

$$\|\mathfrak{f}\|_{1,k}^2 = \sum_{n\geqslant 0} (n+1)^k \|\pi_n \mathfrak{f}\|_1^2$$

Finally, for every local function u, v,

$$\langle\!\langle u, v \rangle\!\rangle_1 = \langle\!\langle \mathfrak{T}u, \mathfrak{T}v \rangle\!\rangle_{1,-1}$$

vanishes for functions u and v of degree 0 or 1. Moreover, for every $n\geqslant 1$ and every finitely supported functions $\mathfrak{f}, \mathfrak{g}\colon \mathscr{E}_{*,n} \to \mathbb{R}$,

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_1 = \tfrac{1}{4} \sum_{x,\, y\neq 0} s(y-x) \sum_{A\in\mathscr{E}_{*,n}} [\mathfrak{g}(A_{x,y}) - \mathfrak{g}(A)]\, [\mathfrak{f}(A_{x,y}) - \mathfrak{f}(A)]$$
$$+ \tfrac{1}{2} \sum_{y\in\mathbb{Z}^d} s(y) \sum_{\substack{A\in\mathscr{E}_{*,n} \\ A\not\ni y}} [\mathfrak{g}(S_y A) - \mathfrak{g}(A)]\, [\mathfrak{f}(S_y A) - \mathfrak{f}(A)]$$

To introduce the dual Hilbert spaces of H_1, $\mathfrak{H}_1$, for a local function u, consider the semi-norm $\|\cdot\|_{-1}$ given by

$$\|u\|_{-1} = \sup_{v} \{2\langle\!\langle u, v\rangle\!\rangle - \langle\!\langle v, v\rangle\!\rangle_1\}$$

where the supremum is taken over all local functions v. Denote by $H_{-1} = H_{-1}(\mathscr{C}, L^s, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$ the Hilbert space generated by the local functions and the semi-norm $\|\cdot\|_{-1}$.

Recall the definition of the spaces $\mathscr{G}_n$ introduced at the beginning of Section 3.1. Since L^s keeps the degree of a function and since the spaces $\mathscr{G}_n$ are orthogonal, for local functions of fixed degree, we may restrict the supremum to local functions of the same degree so that

$$\|f\|_{-1}^2 = \sum_{n\geqslant 1} \|\pi_n f\|_{-1}^2$$

where $\pi_n f$ stands for the projection of f on $\mathscr{G}_n$. Moreover, we will see later in (4.1) that functions of degree 1 which do not vanish in $L^2(\langle\!\langle\cdot,\cdot\rangle\!\rangle)$ have infinite H_{-1} norm.

In the same way, for an integer $n \geqslant 1$ and a finitely supported function $\mathfrak{u}\colon \mathscr{E}_{*,n} \to \mathbb{R}$, let

$$\|\mathfrak{u}\|_{-1}^2 = \sup_{\mathfrak{v}} \{2\langle\mathfrak{u}, \mathfrak{v}\rangle - \langle\mathfrak{v}, \mathfrak{v}\rangle_1\}$$

where the supremum is carried over all finitely supported functions $\mathfrak{v}\colon \mathscr{E}_{*,n} \to \mathbb{R}$. Denote by $\mathfrak{H}_{-1} = \mathfrak{H}_{-1}(\mathscr{E}_{*,n})$ the Hilbert space induced by the finitely supported functions $\mathfrak{u}\colon \mathscr{E}_{*,n} \to \mathbb{R}$ and the semi-norm $\|\cdot\|_{-1}$.

For a integer $k \geqslant -1$, define the $\mathfrak{H}_{-1,k} = \mathfrak{H}_{-1}(\mathscr{E}_*, \mathfrak{L}_s, k)$ norm of a finite supported function $\mathfrak{u}\colon \mathscr{E}_+ \to \mathbb{R}$ by

$$\|\mathfrak{u}\|_{-1,k} = \sup_{\mathfrak{v}} \{2\langle\!\langle\mathfrak{u}, \mathfrak{v}\rangle\!\rangle_{0,k} - \langle\!\langle\mathfrak{v}, (-\mathfrak{L}_s)\,\mathfrak{v}\rangle\!\rangle_{0,k}\}$$

where the supremum is carried over all finitely supported functions $\mathfrak{v}\colon \mathscr{E}_* \to \mathbb{R}$. Denote by $\mathfrak{H}_{-1,k} = \mathfrak{H}_{-1}(\mathscr{E}_*, \mathfrak{L}_s, k)$ the Hilbert space induced by this semi-norm and the space of finite supported functions. Here again, since $\mathfrak{L}_s$ does not change the degrees of a function, for every finitely supported $\mathfrak{u}\colon \mathscr{E}_* \to \mathbb{R}$,

$$\|\mathfrak{u}\|_{-1,k}^2 = \sum_{n\geqslant 1} (n+1)^k \|\pi_n \mathfrak{u}\|_{-1}^2$$

and for any local function u,

$$\|u\|_{-1} = \|\mathfrak{T}\mathfrak{u}\|_{-1,-1}$$

3.3. The Fourier Coefficients of the Generator L

We conclude this section deriving explicit expressions for the generator L on the basis $\{\Psi_A, A \subset \mathbb{Z}^d\}$. A long and elementary computation gives the following dual representation: For every local function $u = \sum_{A \in \mathscr{E}} \mathfrak{u}(A)\, \Psi_A$,

$$Lu = \sum_{A \in \mathscr{E}} (\mathscr{L}_\alpha \mathfrak{u})(A)\, \Psi_A,$$

where $\mathscr{L}_\alpha = \mathscr{L}_s + (1-2\alpha)\, \mathscr{L}_d + \sqrt{\chi(\alpha)}\, (\mathscr{L}_+ + \mathscr{L}_-)$,

$$(\mathscr{L}_d \mathfrak{u})(A) = \sum_{x \in A,\, y \notin A} a(y-x)\{\mathfrak{u}(A_{x,y}) - \mathfrak{u}(A)\}$$

$$(\mathscr{L}_+ \mathfrak{u})(A) = 2 \sum_{x \in A,\, y \in A} a(y-x)\, \mathfrak{u}(A \setminus \{y\})$$

$$(\mathscr{L}_- \mathfrak{u})(A) = 2 \sum_{x \notin A,\, y \notin A} a(y-x)\, \mathfrak{u}(A \cup \{y\})$$

and $\mathscr{L}_s$ is defined by (3.9). Furthermore, for any function $\mathfrak{u}: \mathscr{E} \to \mathbb{R}$, $\mathfrak{T}\mathscr{L}_\alpha \mathfrak{u} = \mathfrak{L}_\alpha \mathfrak{T}\mathfrak{u}$, provided

$$\mathfrak{L}_\alpha = \mathfrak{L}_s + (1-2\alpha)\, \mathfrak{L}_d + \sqrt{\chi(\alpha)}\, \{\mathfrak{L}_+ + \mathfrak{L}_-\}$$

and, for $A \in \mathscr{E}_*$, $\mathfrak{v}: \mathscr{E}_* \to \mathbb{R}$ a finitely supported function,

$$(\mathfrak{L}_d \mathfrak{v})(A) = \sum_{\substack{x \in A,\, y \notin A \\ x,\, y \neq 0}} a(y-x)\{\mathfrak{v}(A_{x,y}) - \mathfrak{v}(A)\} + \sum_{\substack{y \notin A \\ y \neq 0}} a(y)\{\mathfrak{v}(S_y A) - \mathfrak{v}(A)\}$$

$$(\mathfrak{L}_+ \mathfrak{v})(A) = 2 \sum_{x \in A,\, y \in A} a(y-x)\, \mathfrak{v}(A \setminus \{y\})$$

$$+ 2 \sum_{x \in A} a(x)\{\mathfrak{v}(A \setminus \{x\}) - \mathfrak{v}(S_x[A \setminus \{x\}])\}$$

$$(\mathfrak{L}_- \mathfrak{v})(A) = 2 \sum_{\substack{x \notin A,\, y \notin A \\ x,\, y \neq 0}} a(y-x)\, \mathfrak{v}(A \cup \{y\})$$

4. APPROXIMATIONS IN H_{-1}

From this section on, we work in dimension $d \geqslant 3$. All finite constants denoted by C_0 are allowed to depend on the transition probability $p(\cdot)$ and only on $p(\cdot)$.

The main goal of this section is to show that finitely supported functions can be approximated in H_{-1} by finitely supported functions in the image of the operator $\mathfrak{L}_\alpha$. We start proving that all finitely supported functions $\mathfrak{w}: \mathscr{E}_* \to \mathbb{R}$ such that $\mathfrak{w}(\phi) = 0$ are in H_{-1}.

Several estimates on the operators $\mathfrak{L}_d$, $\mathfrak{L}_-$, $\mathfrak{L}_+$ are assumed here and proved in Section 7.

Theorem 4.1. A finitely supported function $\mathfrak{w}: \mathscr{E}_* \to \mathbb{R}$ belongs to $\mathfrak{H}_{-1,k}(\mathscr{E}_*, \mathfrak{L}_s)$ for every $k \geqslant 1$ provided $\mathfrak{w}(\phi) = 0$.

This result states that any local function w in $\bigoplus_{n\geqslant 2} \mathscr{G}_n$ belongs to $H_{-1}(\mathscr{C}, L^s, \langle\!\langle \cdot, \cdot \rangle\!\rangle)$. Notice that we are requiring the degree to be greater or equal to 2. The reason is simple. Any local function f of degree one has H_1 norm equal to 0: if $f = \sum_{x\in\mathbb{Z}^d} c_x \Psi_x$,

$$\begin{aligned}\langle\!\langle f, Lf \rangle\!\rangle &= \langle\!\langle f, L^s f \rangle\!\rangle \\ &= \sum_{x,\, y\in\mathbb{Z}^d} c_x c_y \sum_{z,\, w\in\mathbb{Z}^d} s(w) \langle \Psi_{x+z}, \Psi_{y+w} - \Psi_y \rangle = 0 \end{aligned} \tag{4.1}$$

There are thus two possibilities. Either $\sum_x c_x = 0$, in which case $f = 0$ both in $L^2(\langle\!\langle \cdot, \cdot \rangle\!\rangle)$ and in H_1 or $\sum_x c_x \neq 0$ in which case $f = 0$ in H_1 but not in L^2. In this later circumstance, f does not belong to H_{-1}. Therefore, a function of degree one belongs to H_{-1} only if it vanishes in L^2, i.e., only if $\sum_x c_x = 0$. This explains the restriction on the degree in the previous theorem.

Theorem 4.1 is proved in the same way as Lemma 2.1 in[13]. One has to show that the Green function associated to the generator $\mathfrak{L}_s$ restricted to $\mathscr{E}_{*,n}$, for some $n \geqslant 1$, is finite. This is done by comparing the Green function with the one associated to independent random walks, which is finite because $d \geqslant 3$.

Recall from Remark 3.1 in the previous section that for a finitely supported function $\mathfrak{f}: \mathscr{E} \to \mathbb{R}$, $(\mathfrak{T}\mathfrak{f})(A) = (\mathfrak{T}\mathfrak{f})(S_z A)$ for all z in A. Thus, for $n \geqslant 0$, denote by $\mathscr{I}_n$ the closed subspace of $L^2(\mathscr{E}_{*,n})$ of all functions $\mathfrak{f}$ for which (3.6) holds and let $\mathscr{I} = \bigoplus_{n\geqslant 0} \mathscr{I}_n$.

We are now in a position to state the main result of this section. We have just seen that all finitely supported functions $\mathfrak{w}: \mathscr{E}_* \to \mathbb{R}$ such that

$\mathfrak{w}(\phi) = 0$ belong to H_{-1}. We now prove that these functions can be approximated in H_{-1} by finitely supported functions in the image of the operator $\mathfrak{L}_\alpha$.

Theorem 4.2. Fix a finitely supported function $\mathfrak{w}: \mathscr{E}_* \to \mathbb{R}$ in $\mathscr{I}$ such that $\mathfrak{w}(\phi) = 0$. For each $\varepsilon > 0$ and $k \geqslant -1$, there exists a finitely supported function $\mathfrak{f}: \mathscr{E}_* \to \mathbb{R}$ such that

$$\|\mathfrak{L}_\alpha \mathfrak{f} + \mathfrak{w}\|_{-1, k} \leqslant \varepsilon$$

Moreover, we may take $\mathfrak{f}$ in $\mathscr{I}$ and $\mathfrak{f}(\phi) = 0$.

The proof of this result requires several estimates on the resolvent equation (4.2). The existence of a solution in L^2 of the resolvent equation stated in the next lemma requires a proof because $\mathfrak{L}_\alpha$ is not the generator of a Markov process.

Lemma 4.3. For each $\lambda > 0$, there exists a function $\mathfrak{u}_\lambda$ in $\mathscr{I}$ which solves the resolvent equation

$$\lambda \mathfrak{u}_\lambda - \mathfrak{L}_\alpha \mathfrak{u}_\lambda = \mathfrak{w} \tag{4.2}$$

Proof. For $n \geqslant 1$, let Π_n be the projection on $\bigcup_{j=1}^n \mathscr{E}_{*, j}$: $\Pi_n = \sum_{1 \leqslant j \leqslant n} \pi_j$ and let $\mathfrak{M}_n = \Pi_n(\mathfrak{L}_+ + \mathfrak{L}_-) \Pi_n$. We first prove the existence of a solution in L^2 of the truncated resolvent equation

$$\lambda \mathfrak{u}_{\lambda, n} - \{\mathfrak{L}_s + (1 - 2\alpha) \mathfrak{L}_d + \sqrt{\chi(\alpha)}\, \mathfrak{M}_n\} \mathfrak{u}_{\lambda, n} = \mathfrak{w} \tag{4.3}$$

By Lemma 7.1, the operators $\mathfrak{L}_s$, $\mathfrak{L}_d$, and $\mathfrak{L}_\pm$ are bounded on $L^2(\bigcup_{j=1}^n \mathscr{E}_{*, j})$ for each fixed $n \geqslant 1$. There exists, in particular, a solution for λ large enough. $\mathfrak{L}_s$ is a symmetric negative semi-definite operator while, by Lemma 7.2, $\mathfrak{L}_d$ and $\mathfrak{L}_+ + \mathfrak{L}_-$ are anti-symmetric. Since Π_n is a projection, for any $\mathfrak{f}$ in $L^2(\mathscr{E}_*)$, $\langle \mathfrak{f}, \mathfrak{M}_n \mathfrak{f}\rangle = \langle \mathfrak{f}, \Pi_n(\mathfrak{L}_+ + \mathfrak{L}_-) \Pi_n \mathfrak{f}\rangle = \langle \Pi_n \mathfrak{f}, (\mathfrak{L}_+ + \mathfrak{L}_-) \Pi_n \mathfrak{f}\rangle = 0$, so that $\mathfrak{M}_n$ is also anti-symmetric. Therefore, taking the scalar product on both sides of the previous identity with respect to $\mathfrak{u}_{\lambda, n}$ we obtain by Schwarz inequality that

$$\lambda \|\mathfrak{u}_{\lambda, n}\|_0 \leqslant \|\mathfrak{w}\|_0.$$

Here $\|\cdot\|_0$ stand for the $L^2(\mathscr{E}_*)$ norm. By the proof of Proposition I.2.8 (b) in ref. 12, there exists a solution of (4.3) for every $\lambda > 0$. Moreover, $\mathfrak{u}_{\lambda, n}$ belongs to $\mathscr{I}$ because $\mathfrak{w}$ does.

Up to this point we proved the existence of a solution of the resolvent equation (4.3). The previous estimate shows that the sequence $\{\mathfrak{u}_{\lambda, n}, n \geqslant 1\}$

is uniformly bounded in L^2 for each $\lambda > 0$. Moreover, by Lemma 7.7 and the proof of Lemma 2.5 in ref. 8 or the proof of Theorem 5.1 in ref. 13, since $\mathfrak{w}$ is finitely supported, for every $k \geqslant 1$,

$$\lambda \sum_{j \geqslant 1} j^k \, \|\pi_j \mathfrak{u}_{\lambda, n}\|_0 \leqslant C(k, \mathfrak{w})$$

uniformly over n. Let $\mathfrak{f}$ be a limit point of the sequence $\{\mathfrak{u}_{\lambda, n}, n \geqslant 1\}$. $\mathfrak{f}$ inherits the previous bound and belongs therefore to the domain of the operators $\mathfrak{L}_s$, $\mathfrak{L}_d$, $\mathfrak{L}_\pm$. Furthermore, taking scalar products with finitely supported functions, it is easy to show that any limit point of this sequence is a solution of the resolvent equation (4.2). Finally, $\mathfrak{f}$ belongs to $\mathscr{I}$ because each function $\mathfrak{u}_{\lambda, n}$ belongs and $\mathscr{I}$ is closed. This proves the lemma. ▮

By the proof of Lemma 2.5 in ref. 8 or the proof of Theorem 5.1 in ref. 13 we may deduce from Lemma 7.7 the following bound on the solution $\mathfrak{u}_\lambda$ of the resolvent equation (4.2).

Theorem 4.4. Let $\mathfrak{u}_\lambda$ be the solution of the resolvent equation (4.2). For any positive integer k, there exists a finite constant C_k depending only on k such that

$$\sup_{\lambda \geqslant 0} \{\lambda \, \|\mathfrak{u}_\lambda\|_{0, k}^2 + \|\mathfrak{u}_\lambda\|_{1, k}^2\} \leqslant C_k \, \|\mathfrak{w}\|_{-1, k}^2 \tag{4.4}$$

The following estimate on the asymmetric part of the generator that preserves the degree is needed in the proof of Theorem 4.2.

Lemma 4.5. Let $\mathfrak{u}_\lambda$ be the solution of the resolvent equation (4.2). For any $k \geqslant 1$, there exists a finite constant C_k, depending only on k, such that

$$\|\mathfrak{L}_d \mathfrak{u}_\lambda\|_{-1, k}^2 \leqslant C_k \, \|\mathfrak{w}\|_{-1, k+3}^2.$$

Proof. Let $\mathfrak{w}_1 = \mathfrak{w} + \sqrt{\chi(\alpha)} \, [\mathfrak{L}_+ + \mathfrak{L}_-] \, \mathfrak{u}_\lambda$ so that

$$\lambda \mathfrak{u}_\lambda - \{\mathfrak{L}_s + (1 - 2\alpha) \, \mathfrak{L}_d\} \, \mathfrak{u}_\lambda = \mathfrak{w}_1.$$

By Lemma 7.7 there exists a finite constant C_0 such that

$$\|\pi_n \mathfrak{w}_1\|_{-1}^2 \leqslant 2 \, \|\pi_n \mathfrak{w}\|_{-1}^2 + C_0 n \sum_{j=n-1}^{n+1} \|\pi_j \mathfrak{u}_\lambda\|_1^2 \tag{4.5}$$

for all $n \geqslant 1$.

Notice that the operator $\mathfrak{L}_s + (1-2\alpha)\,\mathfrak{L}_d$ does not change the degree of a function. We may therefore examine equation (4.6) on each set $\mathscr{E}_{*,n}$:

$$\lambda \pi_n \mathfrak{u}_\lambda - \{\mathfrak{L}_s + (1-2\alpha)\,\mathfrak{L}_d\}\,\pi_n \mathfrak{u}_\lambda = \pi_n \mathfrak{w}_1 \tag{4.6}$$

Since n is fixed until estimate (4.10), we omit the operator π_n in the next formulas.

Following Section 6 of ref. 13, we approximate this operator by a convolution operator that can be analyzed through Fourier transforms. Fix $n \geqslant 1$ and let $\mathscr{X}_n = (\mathbb{Z}^d)^n$. We consider a set A in $\mathscr{E}_{*,n}$ as an equivalent class on $n!$ sets of distinct points of $\mathbb{Z}^d_*$. A function $\mathfrak{f}\colon \mathscr{E}_{*,n} \to \mathbb{R}$ can be lifted into a symmetric function $\mathfrak{B}\mathfrak{f}$ on $\mathscr{X}_n$, that vanishes on $\mathscr{X}_n \setminus \mathscr{E}_{n,*}$: $\mathfrak{B}\mathfrak{f}(x_1,..., x_n) = 0$ if $x_i = x_j$ for some $i \neq j$ or if $x_i = 0$ for some $1 \leqslant i \leqslant n$. The operators $\mathfrak{L}_s$, $\mathfrak{L}_d$ can also be extended in a natural way to $\mathscr{X}_n$. Denote by $\{\mathbf{e}_j, 1 \leqslant j \leqslant n\}$ the canonical basis of $\mathbb{R}^n$, by $\mathbf{1}$ the vector $\sum_{1 \leqslant j \leqslant n} \mathbf{e}_j$ and consider on $\mathscr{X}_n$ the operators $\tilde{\mathfrak{L}}_s$, $\tilde{\mathfrak{L}}_d$ defined by

$$\begin{aligned}
(\tilde{\mathfrak{L}}_s \mathfrak{f})(\mathbf{x}) &= \sum_{\substack{1 \leqslant j \leqslant n \\ z \in \mathbb{Z}^d}} s(z)\{\mathfrak{f}(\mathbf{x} + z\mathbf{e}_j) - \mathfrak{f}(\mathbf{x})\} + \sum_{z \in \mathbb{Z}^d} s(z)\{\mathfrak{f}(\mathbf{x} - z\mathbf{1}) - \mathfrak{f}(\mathbf{x})\} \\
(\tilde{\mathfrak{L}}_d \mathfrak{f})(\mathbf{x}) &= \sum_{\substack{1 \leqslant j \leqslant n \\ z \in \mathbb{Z}^d}} a(z)\{\mathfrak{f}(\mathbf{x} + z\mathbf{e}_j) - \mathfrak{f}(\mathbf{x})\} + \sum_{z \in \mathbb{Z}^d} a(z)\{\mathfrak{f}(\mathbf{x} - z\mathbf{1}) - \mathfrak{f}(\mathbf{x})\}
\end{aligned} \tag{4.7}$$

In this formula and below, $\mathbf{x} = (x_1,..., x_n)$ is an element of $\mathscr{X}_n$, so that each x_j belongs to $\mathbb{Z}^d$ and $\mathbf{x} + z\mathbf{e}_j = (x_1,..., x_{j-1}, x_j + z, x_{j+1},..., x_n)$, $\mathbf{x} + z\mathbf{1} = (x_1 + z,..., x_n + z)$.

Denote by $\|\cdot\|_{\mathscr{X}_n, 1}$ the H_1 norm associated to the generator $\tilde{\mathfrak{L}}_s$: for each function $\mathfrak{f}\colon \mathscr{X}_n \to \mathbb{R}$,

$$\|\mathfrak{f}\|^2_{\mathscr{X}_n, 1} = \frac{1}{n!} \sum_{\mathbf{x} \in \mathscr{X}_n} \mathfrak{f}(\mathbf{x})(-\tilde{\mathfrak{L}}_s)\,\mathfrak{f}(\mathbf{x})$$

and denote by $\|\cdot\|_{\mathscr{X}_n, -1}$ its dual norm defined by

$$\|\mathfrak{f}\|^2_{\mathscr{X}_n, -1} = \frac{1}{n!} \sum_{\mathbf{x} \in \mathscr{X}_n} \mathfrak{f}(\mathbf{x})(-\tilde{\mathfrak{L}}_s)^{-1}\,\mathfrak{f}(\mathbf{x})$$

Lifting the resolvent equation (4.6) to $\mathscr{X}_n$ and adding and subtracting $\tilde{\mathfrak{L}}_s \mathfrak{B}\mathfrak{u}_\lambda + (1-2\alpha)\,\tilde{\mathfrak{L}}_d \mathfrak{B}\mathfrak{u}_\lambda$, we obtain that

$$\lambda \mathfrak{B}\mathfrak{u}_\lambda - \{\tilde{\mathfrak{L}}_s + (1-2\alpha)\,\tilde{\mathfrak{L}}_d\}\,\mathfrak{B}\mathfrak{u}_\lambda = \mathfrak{w}_2 \tag{4.8}$$

where

$$\mathfrak{w}_2 = \mathfrak{B}\mathfrak{w}_1 + \{\mathfrak{B}\mathfrak{L}_s - \tilde{\mathfrak{L}}_s\mathfrak{B}\}\ \mathfrak{u}_\lambda + (1-2\alpha)\{\mathfrak{B}\mathfrak{L}_d - \tilde{\mathfrak{L}}_d\mathfrak{B}\}\ \mathfrak{u}_\lambda$$

We claim that $\mathfrak{w}_2$ has finite $H_{-1}(\mathscr{X}_n)$ norm. Indeed, for each $n \geqslant 1$, by (7.5) and Lemma 7.6 below, there exists a finite constant C_0 such that

$$\|\pi_n \mathfrak{B}\mathfrak{w}_1\|^2_{\mathscr{X}_n, -1} \leqslant \|\pi_n \mathfrak{w}_1\|^2_{-1}$$

$$\|\mathfrak{B}\mathfrak{L}_s\pi_n\mathfrak{u}_\lambda - \tilde{\mathfrak{L}}_s\mathfrak{B}\pi_n\mathfrak{u}_\lambda\|^2_{\mathscr{X}_n, -1} \leqslant C_0 n^2\ \|\pi_n\mathfrak{u}_\lambda\|^2_1$$

$$\|\mathfrak{B}\mathfrak{L}_d\pi_n\mathfrak{u}_\lambda - \tilde{\mathfrak{L}}_d\mathfrak{B}\pi_n\mathfrak{u}_\lambda\|^2_{\mathscr{X}_n, -1} \leqslant C_0 n^2\ \|\pi_n\mathfrak{u}_\lambda\|^2_1$$

so that

$$\|\pi_n\mathfrak{w}_2\|^2_{\mathscr{X}_n, -1} \leqslant 2\ \|\pi_n\mathfrak{w}_1\|^2_{-1} + C_0 n^2\ \|\pi_n\mathfrak{u}_\lambda\|^2_1 \tag{4.9}$$

for some finite constant C_0.

It remains to examine the resolvent equation (4.8) through Fourier analysis. Let $\mathbb{T}_{n,d} = [-\pi, \pi]^{nd}$ and denote by $\hat{\mathfrak{u}}_\lambda: (\mathbb{T}^d)^n \to \mathbb{C}$ the Fourier transform of $\mathfrak{B}\mathfrak{u}_\lambda$:

$$\hat{\mathfrak{u}}_\lambda(\mathbf{k}) = \sum_{\mathbf{x} \in \mathscr{X}_n} e^{i\mathbf{x}\cdot\mathbf{k}}\ (\mathfrak{B}\mathfrak{u}_\lambda)(\mathbf{x}).$$

In this formula, $\mathbf{x}\cdot\mathbf{k} = \sum_{1 \leqslant j \leqslant n} x_j \cdot k_j$. It follows from the resolvent equation (4.8) that $\hat{\mathfrak{u}}_\lambda$ is the solution of

$$\lambda\hat{\mathfrak{u}}_\lambda(\mathbf{k}) - \{\hat{\mathfrak{L}}_s(\mathbf{k}) + (1-2\alpha)\ \hat{\mathfrak{L}}_d(\mathbf{k})\}\ \hat{\mathfrak{u}}_\lambda(\mathbf{k}) = \hat{\mathfrak{w}}_2(\mathbf{k})$$

where $\hat{\mathfrak{L}}_s$, $\hat{\mathfrak{L}}_d$ are the functions associated to the operators $\tilde{\mathfrak{L}}_s$, $\tilde{\mathfrak{L}}_d$:

$$-\hat{\mathfrak{L}}_s(\mathbf{k}) = 2 \sum_{\substack{1 \leqslant j \leqslant n \\ z \in \mathbb{Z}^d}} s(z)\{1 - \cos(k_j \cdot z)\} + 2 \sum_{z \in \mathbb{Z}^d} s(z) \left\{1 - \cos\left(\sum_{j=1}^n k_j \cdot z\right)\right\}$$

$$-\hat{\mathfrak{L}}_d(\mathbf{k}) = 2i \sum_{\substack{1 \leqslant j \leqslant n \\ z \in \mathbb{Z}^d}} a(z)\sin(k_j \cdot z) - 2i \sum_{z \in \mathbb{Z}^d} a(z)\sin\left(\sum_{j=1}^n k_j \cdot z\right)$$

The $H_{-1}(\mathscr{X}_n, \hat{\mathfrak{L}}_s)$ norm of a function $\mathfrak{v}: \mathscr{X}_n \to \mathbb{R}$ has a simple and explicit expression in terms of the Fourier transform:

$$\|\mathfrak{v}\|^2_{\mathscr{X}_n, -1} = \frac{-1}{n!(2\pi)^{nd}} \int_{\mathbb{T}_{n,d}} d\mathbf{k}\ |\hat{\mathfrak{v}}(\mathbf{k})|^2\ \frac{1}{\hat{\mathfrak{L}}_s(\mathbf{k})}$$

Since $\mathfrak{B}\mathfrak{u}_\lambda$ is the solution of the resolvent equation (4.8), for every $\lambda > 0$,

$$\|\tilde{\mathfrak{L}}_d \mathfrak{B}\mathfrak{u}_\lambda\|^2_{\mathscr{X}_n, -1} = \frac{-1}{n!(2\pi)^{nd}} \int_{\mathbb{T}_{n,d}} \left| \frac{\hat{\mathfrak{L}}_d(\mathbf{k})}{\lambda - \hat{\mathfrak{L}}_s(\mathbf{k}) - (1-2\alpha)\, \hat{\mathfrak{L}}_d(\mathbf{k})} \right|^2 \frac{|\hat{\mathfrak{w}}_2(\mathbf{k})|^2}{\hat{\mathfrak{L}}_s(\mathbf{k})}\, d\mathbf{k}$$

It follows from the explicit formulas for the functions $\hat{\mathfrak{L}}_s$, $\hat{\mathfrak{L}}_d$ and a Taylor expansion for $|\mathbf{k}|$ small that the previous expression is bounded by

$$\frac{-C_0}{n!(2\pi)^{nd}} \int_{\mathbb{T}_{n,d}} \frac{|\hat{\mathfrak{w}}_2(\mathbf{k})|^2}{\hat{\mathfrak{L}}_s(\mathbf{k})}\, d\mathbf{k} = C_0\, \|\mathfrak{w}_2\|^2_{\mathscr{X}_n, -1}$$

for some finite constant C_0. We have thus proved that

$$\|\tilde{\mathfrak{L}}_d \mathfrak{B}\mathfrak{u}_\lambda\|^2_{\mathscr{X}_n, -1} \leqslant C_0\, \|\mathfrak{w}_2\|^2_{\mathscr{X}_n, -1} \tag{4.10}$$

We may now conclude the proof of the Lemma 4.5. Fix $n \geqslant 1$. By (7.5), by the estimate presented just before (4.9) and by the inequality just derived, there exists a finite constant C_0, which may change from line to line, such that

$$\begin{aligned}
\|\pi_n \mathfrak{L}_d \mathfrak{u}_\lambda\|^2_{-1} = \|\mathfrak{L}_d \pi_n \mathfrak{u}_\lambda\|^2_{-1} &\leqslant C_0 n\, \|\mathfrak{B}\mathfrak{L}_d \pi_n \mathfrak{u}_\lambda\|^2_{\mathscr{X}_n, -1} \\
&\leqslant C_0 n \{ n^2\, \|\pi_n \mathfrak{u}_\lambda\|^2_1 + \|\tilde{\mathfrak{L}}_d \mathfrak{B} \pi_n \mathfrak{u}_\lambda\|^2_{\mathscr{X}_n, -1} \} \\
&\leqslant C_0 \{ n^3\, \|\pi_n \mathfrak{u}_\lambda\|^2_1 + n\, \|\pi_n \mathfrak{w}_2\|^2_{\mathscr{X}_n, -1} \}
\end{aligned}$$

In particular, by (4.9) and (4.5),

$$\begin{aligned}
\|\pi_n \mathfrak{L}_d \mathfrak{u}_\lambda\|^2_{-1} &\leqslant C_0 \{ n^3\, \|\pi_n \mathfrak{u}_\lambda\|^2_1 + n\, \|\pi_n \mathfrak{w}_1\|^2_{\mathscr{X}_n, -1} \} \\
&\leqslant C_0 \left\{ n^3 \sum_{j=n-1}^{n+1} \|\pi_j \mathfrak{u}_\lambda\|^2_1 + n\, \|\pi_n \mathfrak{w}\|^2_{\mathscr{X}_n, -1} \right\}
\end{aligned} \tag{4.11}$$

It remains to recall the definition of the norm $\|\cdot\|_{-1, k}$ and the statement of Theorem 4.4 to conclude the proof. ▌

Notice that the constant C_k, which appears in the statement of Lemma 4.5, depends on k only because the one which appears in Theorem 4.4 depends on k. We have now all elements to prove Theorem 4.2.

Proof of Theorem 4.2. The proof is done in two steps. We first use the resolvent equation (4.2) to obtain a sequence $\{\mathfrak{v}_j;\, j \geqslant 1\}$ of functions in $L^2(\mathscr{E}_*, k)$ such that

$$\lim_{j \to \infty} \|\mathfrak{L}_\alpha \mathfrak{v}_j + \mathfrak{w}\|_{-1, k} = 0$$

The sequence $\mathfrak{v}_j$ is obtained through convex combinations (in λ) of the solutions $\mathfrak{u}_\lambda$ of the resolvent equation (4.2). Details can be found at the end of the proof of Lemma 2.1 in ref. 8 or at the beginning of the proof of Lemma 2.8 in ref. 10.

At this point, it remains to show the existence, for each fixed j and $\varepsilon > 0$, of a finitely supported function $\mathfrak{f}: \mathscr{E}_* \to \mathbb{R}$ such that $\|\mathfrak{L}_\alpha f - \mathfrak{L}_\alpha \mathfrak{v}_j\|_{-1,k} \leqslant \varepsilon$. To prove the existence of such function $\mathfrak{f}$, assume that $\mathfrak{f}$ vanishes on $\bigcup_{j \geqslant n} \mathscr{E}_{*,j}$ and recall the decomposition of the operator $\mathfrak{L}_\alpha$ to deduce that

$$\begin{aligned}
&\|\mathfrak{L}_\alpha \mathfrak{f} - \mathfrak{L}_\alpha \mathfrak{v}_j\|_{-1,k} \\
&\quad \leqslant \|\mathfrak{L}_+(\mathfrak{f} - \Pi_n \mathfrak{v}_j)\|_{-1,k} + \|\mathfrak{L}_-(\mathfrak{f} - \Pi_n \mathfrak{v}_j)\|_{-1,k} \\
&\qquad + \|\mathfrak{L}_d(\mathfrak{f} - \Pi_n \mathfrak{v}_j)\|_{-1,k} + \|\mathfrak{L}_s(\mathfrak{f} - \Pi_n \mathfrak{v}_j)\|_{-1,k} + \|\mathfrak{L}_\alpha(I - \Pi_n)\, \mathfrak{v}_j)\|_{-1,k}
\end{aligned} \tag{4.12}$$

where $\Pi_n = \sum_{j \leqslant n} \pi_j$ and I is the identity. We estimate each term on the right hand side separately. By Lemma 7.7, the first term on the right hand side is such that

$$\begin{aligned}
\|\mathfrak{L}_+(\mathfrak{f} - \Pi_n \mathfrak{v}_j)\|_{-1,k}^2 &= \sum_{\ell=0}^{n} (\ell+1)^k \|\mathfrak{L}_+ \pi_\ell \{\mathfrak{f} - \mathfrak{v}_j\}\|_{-1}^2 \\
&\leqslant C_0 \sum_{\ell=1}^{n+1} \ell^{k+1} \|\pi_\ell \mathfrak{f} - \pi_\ell \mathfrak{v}_j\|_1^2 \leqslant C_0 \sum_{\ell=1}^{n+1} \ell^{k+2} \|\pi_\ell \mathfrak{f} - \pi_\ell \mathfrak{v}_j\|_0^2
\end{aligned}$$

The second term on the right hand side of (4.12) is estimated in the same way. By Lemma 7.4, the third one is such that

$$\|\mathfrak{L}_d(\mathfrak{f} - \Pi_n \mathfrak{v}_j)\|_{-1,k}^2 = \sum_{\ell=1}^{n} \ell^k \|\mathfrak{L}_d \pi_\ell (\mathfrak{f} - \mathfrak{v}_j)\|_{-1}^2 \leqslant C_0 \sum_{\ell=1}^{n} \ell^{k+1} \|\pi_\ell (\mathfrak{f} - \mathfrak{v}_j)\|_0^2$$

for some finite constant C_0. The fourth member on the right hand side of (4.12) is easily estimated by exactly the same arguments and by using again Lemma 7.4.

Finally, since $\mathfrak{v}_j$ is a convex combination of the solutions of the resolvent equation (4.2), by (4.11),

$$\begin{aligned}
\|\mathfrak{L}_\alpha(I - \Pi_n)\, \mathfrak{v}_j)\|_{-1,k}^2 &= \sum_{\ell > n} \ell^k \|\mathfrak{L}_\alpha \pi_\ell \mathfrak{v}_j\|_{-1}^2 \\
&\leqslant \sum_{\ell \geqslant n} \ell^{k+3} \|\pi_\ell \mathfrak{v}_j\|_1^2 + \sum_{\ell \geqslant n} \ell^{k+1} \|\pi_\ell \mathfrak{w}\|_{-1}^2
\end{aligned}$$

Now, for $\varepsilon > 0$ fixed, since $\mathfrak{w}$ is finitely supported, by Theorem 4.4 and Theorem 4.1 there exists $n_0 > 0$ large enough for the last quantity to be bounded by ε. For this fixed n_0, find a finitely supported function $\mathfrak{f}: \bigcup_{n \leqslant n_0} \mathscr{E}_{*,n} \to \mathbb{R}$ for which all previous expressions are bounded by ε, which is possible because $\mathfrak{v}_j$ belongs to $L^2(\mathscr{E}_*, k)$.

It remains to check that we may take $\mathfrak{f}$ in $\mathscr{I}$ with $\mathfrak{f}(\phi) = 0$. The first property follows from the fact, easy to verify, that the operators $\mathfrak{L}_s$, $\mathfrak{L}_d$, $\mathfrak{L}_+$, $\mathfrak{L}_-$ map the closed subspace $\mathscr{I}$ of $L^2(\mathscr{E}_*)$ into $\mathscr{I}$. In particular, the solutions of the resolvent equations, as well as their convex combinations, belong to $\mathscr{I}$ so that $\mathfrak{f}$ can be taken in $\mathscr{I}$.

The second requirement follows from the fact that $(\mathfrak{L}_* \mathfrak{f})(\phi) = 0$ for any finitely supported function $\mathfrak{f}$ in $\mathscr{I}$, where $\mathfrak{L}_*$ stands for any of the four operators $\mathfrak{L}_s$, $\mathfrak{L}_d$, $\mathfrak{L}_+$, $\mathfrak{L}_-$ and from the fact that $(\mathfrak{L}_+ \mathfrak{f})(\{x\}) = 0$ for all x in $\mathbb{Z}_*^d$. These two properties show that we may set the value of $\mathfrak{f}$ at ϕ to be 0 without changing $\mathfrak{L}_\alpha \mathfrak{f}$.

The special features of the operators $\mathfrak{L}_*$ just used are proved in the beginning of Section 7. This concludes the proof. ∎

5. THE FLUCTUATION-DISSIPATION THEOREM

We consider in this section the general asymmetric simple exclusion in dimension $d \geqslant 3$. Here again, all finite constants C_0 which appear in the statement of the theorems may depend only on the transition probability $p(\cdot)$.

We prove in this section a fluctuation–dissipation theorem, Theorem 5.3 below, also called the Boltzmann–Gibbs principle. It allows the replacement of a local function w in $\bigoplus_{n \geqslant 2} \mathscr{G}_n$ by the sum of gradients $\eta(x+e_j) - \eta(x)$ and local functions in the range of the generator. This result is the main step in the proof of Gaussian fluctuations of the empirical measure around the hydrodynamic limit (cf. ref. 4).

Denote by $\mathbb{E}_{\nu_\alpha}$ the expectation on the path space $D(\mathbb{R}_+, \mathscr{X})$ corresponding to the Markov process with generator given by (2.1) and starting from the stationary measure ν_α. The first result states that a local function f in $H_{-1}(L^s, \langle\!\langle \cdot, \cdot \rangle\!\rangle)$ has a finite space-time variance in the diffusive scaling:

Theorem 5.1. Fix a smooth function $G: \mathbb{R}^d \to \mathbb{R}$ with compact support, a local function f of degree $n \geqslant 2$, $T > 0$ and a vector v in $\mathbb{R}^d$. There exists a finite constant C_0 such that

$$\limsup_{\epsilon \to 0} \mathbb{E}_{\nu_\alpha} \left[\sup_{0 \leqslant t \leqslant T} \left(\epsilon^{d/2+1} \int_0^{t\epsilon^{-2}} \sum_{x \in \mathbb{Z}^d} G(\epsilon(x - rv)) \, f(\tau_x \eta_r) \, dr \right)^2 \right]$$
$$\leqslant C_0 T \, \|G\|_{L^2}^2 \, \|\mathfrak{T}\mathfrak{f}\|_{-1}^2$$

Proof. Set $G^\epsilon_r(x) = G(\epsilon(x - rv))$ and recall Lemma 4.3 in ref. 2 to obtain that

$$\mathbb{E}_{\nu_\alpha}\left[\sup_{0\leqslant t\leqslant T}\left(\epsilon^{d/2+1}\int_0^{t\epsilon^{-2}} dr \sum_{x\in\mathbb{Z}^d} G^\epsilon_r(x)\, f(\tau_x\eta_r)\, dr\right)^2\right]$$

$$\leqslant 14\int_0^T dr\,\epsilon^d\left\langle \sum_{x\in\mathbb{Z}^d} G^\epsilon_r(x)\,\tau_x f, (-L^s)^{-1}\sum_{x\in\mathbb{Z}^d} G^\epsilon_r(x)\,\tau_x f\right\rangle \quad (5.1)$$

Fix a local function g of degree $n\geqslant 2$. Let $\mathscr{L}_s$ be the generator defined in (3.9) and $\mathfrak{g}$ be the Fourier coefficient of g. Then,

$$\langle g, (-L^s)^{-1}\, g\rangle = \langle \mathfrak{g}, (-\mathscr{L}_s)^{-1}\,\mathfrak{g}\rangle$$

where the last scalar product is on $\mathscr{E}_n$. Recall from Section 4 that $\mathscr{X}_n = (\mathbb{Z}^d)^n$. The proof of Lemma 7.5 shows that the previous expression is bounded by

$$\frac{C_0}{(n-1)!}\,\langle \mathfrak{B}\mathfrak{g}, (-\widetilde{\mathscr{L}}{}^s_0)^{-1}\,\mathfrak{B}\mathfrak{g}\rangle_{\mathscr{X}_n}$$

for some finite constant C_0, where $\widetilde{\mathscr{L}}{}^s_0$ stands for the generator of n independent random walks on $\mathbb{Z}^d$ with transition probability $s(\cdot)$ and $\mathfrak{B}\mathfrak{g}$ for the extension of $\mathfrak{g}$ to $\mathscr{X}_n$. Denote by $\mathfrak{G}_n(\cdot,\cdot)$ the Green function associated to the generator $\widetilde{\mathscr{L}}{}^s_0$ restricted to $\mathscr{X}_n$. The previous expression can be written as

$$\frac{C_0}{(n-1)!}\sum_{\mathbf{x},\mathbf{y}\in\mathscr{X}_n} (\mathfrak{B}\mathfrak{g})(\mathbf{x})\,\mathfrak{G}_n(\mathbf{x},\mathbf{y})(\mathfrak{B}\mathfrak{g})(\mathbf{y})$$

In particular, for $g = \sum_{x\in\mathbb{Z}^d} G^\epsilon_r(x)\,\tau_x f$, the right hand side of (5.1) is bounded by

$$\frac{C_0}{(n-1)!}\int_0^T dr\,\epsilon^d \sum_{z,w\in\mathbb{Z}^d} G^\epsilon_r(z)\,G^\epsilon_r(w)\sum_{\mathbf{x},\mathbf{y}\in\mathscr{X}_n}\mathfrak{B}\mathfrak{f}(\mathbf{x}+z\mathbf{1})\,\mathfrak{G}_n(\mathbf{x},\mathbf{y})\,\mathfrak{B}\mathfrak{f}(\mathbf{y}+w\mathbf{1})$$

A change of variables permits to rewrite the expression inside the integral as

$$\epsilon^d\sum_{z,w\in\mathbb{Z}^d} G^\epsilon_r(z)\,G^\epsilon_r(z-w)\sum_{\mathbf{x},\mathbf{y}\in\mathscr{X}_n}\mathfrak{B}\mathfrak{f}(\mathbf{x})\,\mathfrak{G}_n(\mathbf{x},\mathbf{y}+w\mathbf{1})\,\mathfrak{B}\mathfrak{f}(\mathbf{y})$$

Since f is a local function, $\mathfrak{f}$ has finite support. In particular, the sums over $\mathbf{x}$, $\mathbf{y}$ are carried over finite sets. On the other hand, replacing $G_r^\epsilon(z-w)$ by $G_r^\epsilon(z)$, we write the previous expression as

$$\epsilon^d \sum_{z \in \mathbb{Z}^d} G_r^\epsilon(z)^2 \sum_{\mathbf{x},\mathbf{y} \in \mathcal{X}_n} \mathfrak{B}\mathfrak{f}(\mathbf{x}) \sum_{w \in \mathbb{Z}^d} \mathfrak{G}_n(\mathbf{x}, \mathbf{y}+w\mathbf{1})\, \mathfrak{B}\mathfrak{f}(\mathbf{y})$$

$$+\epsilon^d \sum_{z,w \in \mathbb{Z}^d} G_r^\epsilon(z)\{G_r^\epsilon(z-w)-G_r^\epsilon(z)\} \sum_{\mathbf{x},\mathbf{y} \in \mathcal{X}_n} \mathfrak{B}\mathfrak{f}(\mathbf{x})\, \mathfrak{G}_n(\mathbf{x}, \mathbf{y}+w\mathbf{1})\, \mathfrak{B}\mathfrak{f}(\mathbf{y})$$

We claim that the second term is bounded above by $C(f, G)\,\epsilon^{1/2}$ for some finite constant depending on f and G. Indeed, since G has a bounded derivative and $\mathfrak{B}\mathfrak{f}$ a finite support, the second line is less than or equal to

$$C(\|\nabla G\|_\infty, f)\,\epsilon^{1/2} \sup_{\mathbf{x} \in \mathcal{X}_{\mathfrak{f}}^-} \sum_{w \in \mathbb{Z}^d} |w|^{1/2}\, \mathfrak{G}_n(0, \mathbf{x}+w\mathbf{1}). \tag{5.2}$$

In this formula, $\mathcal{X}_{\mathfrak{f}}^-$ is the set of all sites which can be written as difference of two points in the support of $\mathfrak{B}\mathfrak{f}$: $\mathcal{X}_{\mathfrak{f}}^- = \{\mathbf{y}-\mathbf{x}, \mathfrak{B}\mathfrak{f}(\mathbf{x})\, \mathfrak{B}\mathfrak{f}(\mathbf{y}) \neq 0\}$. Since $\mathfrak{G}_n(0, \mathbf{z})$ is the Green function of n independent random walks on $\mathbb{Z}^d$, it decays as $|\mathbf{z}|^{2-nd}$. The sum over w is thus finite, uniformly over $\mathbf{x}$, as soon as $(n-1)\,d > 5/2$, inequality which is fulfilled because we are assuming $d \geqslant 3$ and $n \geqslant 2$. This proves (5.2).

Collecting all previous estimates we obtain that the expectation which appears in the statement of the lemma is bounded above by

$$\frac{C_0}{(n-1)!} \int_0^T dr\, \epsilon^d \sum_{z \in \mathbb{Z}^d} G_r^\epsilon(z)^2 \sum_{\mathbf{x},\mathbf{y} \in \mathcal{X}_n} \mathfrak{B}\mathfrak{f}(\mathbf{x}) \sum_{w \in \mathbb{Z}^d} \mathfrak{G}_n(\mathbf{x}, \mathbf{y}+w\mathbf{1})\, \mathfrak{B}\mathfrak{f}(\mathbf{y})$$

plus a remainder which converges to 0 as $\epsilon \downarrow 0$. Denote by $\tilde{\mathfrak{G}}_n(\cdot,\cdot)$ the Green function associated to the generator $\tilde{\mathfrak{L}}_s$ defined in (4.7) and restricted to $\mathcal{X}_n$. An elementary computation shows that

$$\sum_{\mathbf{x},\mathbf{y} \in \mathcal{X}_n} \mathfrak{B}\mathfrak{f}(\mathbf{x}) \sum_{w \in \mathbb{Z}^d} \mathfrak{G}_n(\mathbf{x}, \mathbf{y}+w\mathbf{1})\, \mathfrak{B}\mathfrak{f}(\mathbf{y}) = \sum_{\mathbf{x},\mathbf{y} \in \mathcal{X}_{n-1}} \mathfrak{B}\bar{\mathfrak{f}}(\mathbf{x})\, \tilde{\mathfrak{G}}_{n-1}(\mathbf{x}, \mathbf{y})\, \mathfrak{B}\bar{\mathfrak{f}}(\mathbf{y})$$
$$= (n-1)!\, \|\pi_{n-1} \mathfrak{B}\bar{\mathfrak{f}}\,\|_{-1, \mathcal{X}_{n-1}}^2$$

On the right hand side, $\bar{\mathfrak{f}}$ stands for $\mathfrak{T}\mathfrak{f}$. Notice that n is replaced by $n-1$ in the right hand side because we are fixing a particle at the origin. The last time integral is thus equal to

$$C_0 \int_0^T dr\, \epsilon^d \sum_{z \in \mathbb{Z}^d} G_r^\epsilon(z)^2\, \|\mathfrak{B}\bar{\mathfrak{f}}\|_{\mathcal{X}_n, -1}^2 \leqslant C_0 \int_0^T dr\, \epsilon^d \sum_{z \in \mathbb{Z}^d} G_r^\epsilon(z)^2\, \|\bar{\mathfrak{f}}\|_{-1}^2$$

where in the last step we used Lemma 7.5. As $\epsilon \downarrow 0$ this integral converges to

$$C_0 T \, \|G\|_{L^2}^2 \, \|\bar{\mathfrak{f}}\|_{-1}^2$$

which concludes the proof of the lemma. ∎

Corollary 5.2. Fix a smooth function $G: \mathbb{R}^d \to \mathbb{R}$ with compact support, a local function f in $\bigoplus_{n \geqslant 2} \mathscr{G}_n$, $T > 0$ and a vector v in $\mathbb{R}^d$. There exists a finite constant C_0 such that

$$\limsup_{\epsilon \to 0} \mathbb{E}_{\nu_\alpha} \left[\sup_{0 \leqslant t \leqslant T} \left| \epsilon^{d/2+1} \int_0^{t\epsilon^{-2}} \sum_{x \in \mathbb{Z}^d} G(\epsilon(x - rv)) \, f(\tau_x \eta_r) \, dr \right|^2 \right]$$
$$\leqslant C_0 T \, \|G\|_{L^2}^2 \, \|\mathfrak{T}\mathfrak{f}\|_{-1,0}^2$$

Proof. Since the symmetric part of the generator does not change the degree of local functions, for a local function g in $\bigoplus_{n \geqslant 2} \mathscr{G}_n$,

$$\langle g, (-L^s)^{-1} g \rangle = \sum_{n \geqslant 2} \langle \pi_n g, (-L^s)^{-1} \, \pi_n g \rangle = \sum_{n \geqslant 2} \langle \pi_n \mathfrak{g}, (-\mathscr{L}_s)^{-1} \, \pi_n \mathfrak{g} \rangle$$

and we may proceed as in the proof of Theorem 5.1. ∎

Fix a local function w in $\bigoplus_{n \geqslant 2} \mathscr{G}_n$ and denote by $\mathfrak{w}$ its Fourier coefficients and by $\bar{\mathfrak{w}}$ the finitely supported function $\mathfrak{T}\mathfrak{w}: \mathscr{E}_* \to \mathbb{R}$. For each $\lambda > 0$, let $\mathfrak{u}_\lambda$ be the solution of the resolvent equation

$$\lambda \mathfrak{u}_\lambda - \mathfrak{L}_\alpha \mathfrak{u}_\lambda = \bar{\mathfrak{w}}$$

We proved in Section 4 the existence of the solution $\mathfrak{u}_\lambda$ of the resolvent equation and some of its properties. In the next section, among several other properties, we show the existence of a subsequence λ_k for which the sequence $\mathfrak{u}_{\lambda_k}(\alpha, z)$ converges, as $k \uparrow \infty$, uniformly in α in $[0, 1]$, to some limit, denoted by $D_z(\alpha)$:

$$D_z(\alpha) = \lim_{k \to \infty} \mathfrak{u}_{\lambda_k}(\alpha, \{z\}) \tag{5.3}$$

for each z in $\mathbb{Z}_*^d$.

Theorem 5.3. Fix a local function w in $\bigoplus_{n \geqslant 2} \mathscr{G}_n$ and a smooth function $G: \mathbb{R}^d \to \mathbb{R}$ with compact support. There exists a sequence of local functions u_m such that

$$\limsup_{m \to \infty} \limsup_{\epsilon \to 0} \mathbb{E}_{\nu_\alpha} \left[\sup_{0 \leqslant t \leqslant T} \left(\epsilon^{d/2+1} \int_0^{t\epsilon^{-2}} \sum_{x \in \mathbb{Z}^d} G(\epsilon x) \, \tau_x W_m(\eta_s) \, ds \right)^2 \right] = 0$$

where

$$W_m(\eta) = W_{\alpha, m}(\eta) = w - Lu_m + \sqrt{\chi(\alpha)} \sum_{z \in \mathbb{Z}^d} a(z)\, D_z(\alpha)\{\Psi_z - \Psi_0\}$$

We will refer to Theorem 5.3 as the *fluctuation-dissipation theorem*. It will be the basic ingredient to study the equilibrium fluctuations for the density field.

Proof. Recall that $\bar{\mathfrak{w}} = \mathfrak{T}\mathfrak{w}$. Since w belongs to $\bigoplus_{n \geqslant 2} \mathcal{G}_n$, by Theorem 4.2 with $k = 0$, there exists a sequence of finitely supported functions $\mathfrak{v}_m \colon \mathcal{E}_* \to \mathbb{R}$ such that $\mathfrak{v}_m(\phi) = 0$, $\mathfrak{v}_m$ belongs to $\mathcal{I}$ and

$$\lim_{m \to \infty} \|\bar{\mathfrak{w}} - \mathfrak{L}_\alpha \mathfrak{v}_m\|_{-1} = 0$$

Since $\mathfrak{v}_m$ satisfies (3.6), in view of (3.7), there exists a finitely supported function $\mathfrak{u}_m \colon \mathcal{E} \to \mathbb{R}$ such that $\mathfrak{T}\mathfrak{u}_m = \mathfrak{v}_m$. Moreover, $\mathfrak{u}_m(A) \neq 0$ only if A contains the origin and $\mathfrak{u}_m(\{0\}) = 0$ because $\mathfrak{v}_m(\phi) = 0$. Let u_m be the local function defined by $u_m = \sum_{A \in \mathcal{E}} \mathfrak{u}_m(A)\, \Psi_A$ and let $\hat{W}_m$ be the local function in $\bigoplus_{n \geqslant 2} \mathcal{G}_n$ defined by

$$\hat{W}_m = w - \{Lu_m - \pi_1 Lu_m\}$$

By Corollary 5.2,

$$\limsup_{\epsilon \to 0} \mathbb{E}_{\nu_\alpha}\left[\sup_{0 \leqslant t \leqslant T} \left(\epsilon^{d/2+1} \int_0^{t\epsilon^{-2}} \sum_{x \in \mathbb{Z}^d} G(\epsilon x)\, \tau_x \hat{W}_m(\eta_s)\, ds\right)^2\right] \leqslant C_0 T\, \|G\|_{L^2}^2\, \|\mathfrak{T}\hat{W}_m\|_{-1}^2 \tag{5.4}$$

for some finite constant C_0. An elementary computation gives that

$$\begin{aligned}\pi_1(Lu) = &- \sum_{x, y \in \mathbb{Z}^d} s(y-x)\{\mathfrak{u}(y) - \mathfrak{u}(x)\}\{\Psi_y - \Psi_x\} \\ &- \frac{1-2\alpha}{2} \sum_{x, y \in \mathbb{Z}^d} a(y-x)\{\mathfrak{u}(y) + \mathfrak{u}(x)\}\{\Psi_y - \Psi_x\} \\ &+ \sqrt{\chi(\alpha)} \sum_{x, y \in \mathbb{Z}^d} a(y-x)\, \mathfrak{u}(x, y)\{\Psi_y - \Psi_x\}\end{aligned}$$

Since $\mathfrak{u}_m(\{x\}) = 0$ for every x in $\mathbb{Z}^d$, $\mathfrak{u}_m(A) \neq 0$ only if A contains the origin, $\mathfrak{u}_m(A) = |A|^{-1}\, \mathfrak{v}_m(A \setminus \{0\})$, we have that

$$\pi_1(Lu) = \sqrt{\chi(\alpha)} \sum_{x \in \mathbb{Z}^d} a(x)\, \mathfrak{v}_m(x)\{\Psi_x - \Psi_0\}$$

which vanishes in $L^2(\chi, \langle\!\langle \cdot, \cdot \rangle\!\rangle)$ so that $\mathfrak{I}\pi_1(Lu) = 0$. In particular, $\mathfrak{I}\hat{W}_m = \bar{\mathfrak{w}} - \mathfrak{L}_\alpha \mathfrak{v}_m$ and the right hand side of (5.4) is bounded above by

$$C_0 T \, \|G\|_{L^2}^2 \, \|\bar{\mathfrak{w}} - \mathfrak{L}_\alpha \mathfrak{v}_m\|_{-1}^2$$

Since this expression vanishes as $m \uparrow \infty$, all we need to prove is that

$$\limsup_{m \to \infty} \limsup_{\epsilon \to 0} \mathbb{E}_{\nu_\alpha} \left[\sup_{0 \leqslant t \leqslant T} \left(\epsilon^{d/2+1} \int_0^{t\epsilon^{-2}} \sum_{x \in \mathbb{Z}^d} G(\epsilon x) \, \tau_x \tilde{W}_m(\eta_s) \, ds \right)^2 \right] = 0$$

where

$$\tilde{W}_m = \pi_1 L u_m - \sqrt{\chi(\alpha)} \sum_{z \in \mathbb{Z}^d} a(z) \, D_z(\alpha) \{\Psi_z - \Psi_0\}$$

By the explicit formula for $\pi_1 L u_m$ obtained above,

$$\tilde{W}_m = \sqrt{\chi(\alpha)} \sum_{x \in \mathbb{Z}^d} a(x) \{\mathfrak{v}_m(x) - D_x(\alpha)\} \{\Psi_x - \Psi_0\}$$

Since ν_α is a stationary state, by Schwarz inequality, the previous expectation is bounded above by

$$\epsilon^{d-2} T^2 E_{\nu_\alpha} \left[\left(\sum_{x \in \mathbb{Z}^d} G(\epsilon x) \, \tau_x \tilde{W}_m(\eta) \right)^2 \right]$$

A change of variables gives that this expression is equal to

$$\epsilon^{d-2} \chi(\alpha) \, T^2 \sum_{x \in \mathbb{Z}^d} \left(\sum_{y \in \mathbb{Z}^d} a(y) \{G(\epsilon[x-y]) - G(\epsilon x)\} \{\mathfrak{v}_m(y) - D_y(\alpha)\} \right)^2$$

A Taylor expansion shows that this expression converges, as $\epsilon \downarrow 0$, to

$$\chi(\alpha) \, T^2 \int_{\mathbb{R}^d} da \, \{(\nabla G)(a) \cdot R_m\}^2$$

where $R_m = (R_m^1, ..., R_m^d)$, $R_m^j = \sum_{y \in \mathbb{Z}^d} a(y) \, y_j \{\mathfrak{v}_m(y) - D_y(\alpha)\}$. By the definition of $D_y(\alpha)$, $R_m^j \to 0$ as $m \uparrow \infty$ because $\mathfrak{v}_m$ is constructed as convex combinations of $\mathfrak{u}_{\lambda_k}$, the solution of the resolvent equation (4.2). The integral therefore tends to 0. ∎

6. REGULARITY OF VISCOSITY COEFFICIENTS

Fix a local function w in $\bigoplus_{n\geqslant 2}\mathcal{G}_n$ and denote by $\mathfrak{w}$ its Fourier coefficients and by $\bar{\mathfrak{w}}$ the finitely supported function $\mathfrak{T}\mathfrak{w}\colon \mathscr{E}_* \to \mathbb{R}$. For each $\lambda > 0$, let $\mathfrak{u}_\lambda$ be the solution of the resolvent equation

$$\lambda \mathfrak{u}_\lambda - \mathfrak{L}_\alpha \mathfrak{u}_\lambda = \bar{\mathfrak{w}}$$

We proved in Section 4 the existence of the solution $\mathfrak{u}_\lambda$ of the resolvent equation. We prove in this section the existence of a subsequence λ_k for which $\mathfrak{u}_{\lambda_k}(\cdot, z)$ converges uniformly in $[0, 1]$, as well as all its derivatives, to a smooth function $D_z(\cdot)$.

The proof is very close to the one presented in ref. 11 for the self diffusion in the symmetric case, so we only show here the main points.

We want to show that there exists a subsequence $\lambda_k \downarrow 0$, such that, for each z such that $|a(z)| > 0$, $\mathfrak{u}_{\lambda_k}(\alpha, \{z\})$ converges uniformly in α to a smooth function. To prove the existence of such subsequence it is enough to show that $\mathfrak{u}_\lambda(\alpha, \{z\})$ are smooth function of α for each $\lambda > 0$ and each z, and

$$\sup_{\lambda>0}\ \sup_{0\leqslant\alpha\leqslant 1} |\mathfrak{u}_\lambda^{(j)}(\alpha, \{z\})| < \infty$$

where $\mathfrak{u}_\lambda^{(j)}(\alpha, \{z\})$ stands for the jth derivative of $\mathfrak{u}_\lambda$.

By Lemma 3.1 of ref. 13, we have the bound

$$|\mathfrak{u}_\lambda^{(j)}(\alpha, \{z\})| \leqslant C_0\, \|\mathfrak{u}_\lambda^{(j)}(\alpha, \cdot)\|_1$$

for some finite constant C_0 depending only on $p(\cdot)$. With the iterated use of Theorem 4.4, we will prove that, for any k and j,

$$\sup_{\lambda>0}\ \sup_{0\leqslant\alpha\leqslant 1} \|\mathfrak{u}_\lambda^{(j)}(\alpha, \cdot)\|_{1,k} < \infty$$

Since the coefficients of $\mathfrak{L}_\alpha$ are not smooth at the boundary of $[0, 1]$, we reparameterize by $\alpha = \sin^2 t$, $t \in [0, 2\pi]$, as done in ref. 11. We obtain

$$\mathfrak{L}(t) = \mathfrak{L}_s + (\cos^2 t - \sin^2 t)\, \mathfrak{L}_d + (\sin t \cos t)\{\mathfrak{L}_+ + \mathfrak{L}_-\}$$

and then we consider the equation

$$\lambda \mathfrak{v}_\lambda(t) - \mathfrak{L}(t)\, \mathfrak{v}_\lambda(t) = \mathfrak{w}$$

Since $\mathfrak{w}$ does not depends on α, we have $\mathfrak{u}_\lambda(\alpha(t)) = v_\lambda(t)$. So if we prove that $v_\lambda^{(j)}(t)$ are uniformly bounded in the $\|\cdot\|_{1,k}$ norm, we obtain the boundedness in the same norm for $\mathfrak{u}_\lambda(\alpha)$ for α in the interior of $[0, 1]$.

The extra argument to extend this smoothness up to the boundary is identical to the one used in ref. 11 (see the end of the proof of Theorem 5.1 in there).

Differentiating formally $\mathfrak{L}(t)$ in t, we obtain

$$\mathfrak{L}'(t) = -4(\sin t \cos t)\ \mathfrak{L}_d + (\cos^2 t - \sin^2 t)\{\mathfrak{L}_+ + \mathfrak{L}_-\}$$

By Lemma 5.2 of ref. 11, $\mathfrak{v}_\lambda(t)$ is differentiable in t, and its derivative $\mathfrak{v}'_\lambda(t)$ satisfies

$$\lambda \mathfrak{v}'_\lambda(t) - \mathfrak{L}(t)\ \mathfrak{v}'_\lambda(t) = \mathfrak{L}'(t)\ \mathfrak{v}_\lambda(t)$$

As a consequence of Lemma 7.7, Lemma 4.5 and the explicit form of $\mathfrak{L}(t)$, there exists a constant C_k depending only on k

$$\|\mathfrak{L}'(t)\ \mathfrak{v}_\lambda(t)\|_{-1,k} \leqslant C_k\ \|\bar{\mathfrak{w}}\|_{-1,k+3}$$

Then if we apply Theorem 4.4 to Eq. (6.1), we obtain the bound $\|\mathfrak{v}'_\lambda(t)\|_{-1,k}$ uniform in t and λ. The argument can be iterated exactly as done in ref. 11, obtaining similar bounds for all the derivatives $v_\lambda^{(j)}(t)$.

7. ESTIMATES ON THE OPERATORS $\mathfrak{L}_d$, $\mathfrak{L}_+$, $\mathfrak{L}_-$

We prove in this section some elementary identities or estimates involving the operators $\mathfrak{L}_d$, $\mathfrak{L}_+$, $\mathfrak{L}_-$. Recall that all functions $\mathfrak{f}: \mathscr{E}_* \to \mathbb{R}$ which come from a local function f through the transformation $\mathfrak{T}$ of the Fourier coefficients of f are such that $\mathfrak{f}(S_z A) = \mathfrak{f}(A)$ for all z in A. Recall also the definition of the spaces $\mathscr{I}_n$ given just before the statement of Theorem 4.2.

A simple computation shows that the space $\mathscr{I}$ is left invariant by the operators $\mathfrak{L}_s$, $\mathfrak{L}_d$, $\mathfrak{L}_+$, $\mathfrak{L}_-$: For every $n \geqslant 1$ and every $\mathfrak{f}: \mathscr{I}_n \to \mathbb{R}$,

$$\mathfrak{L}_s \mathfrak{f} \in \mathscr{I}_n, \qquad \mathfrak{L}_d \mathfrak{f} \in \mathscr{I}_n, \qquad \mathfrak{L}_- \mathfrak{f} \in \mathscr{I}_{n-1}, \qquad \mathfrak{L}_+ \mathfrak{f} \in \mathscr{I}_{n+1} \tag{7.1}$$

This claim can be proved in two different ways. Either by a direct computation or by reconstructing a local function f from $\mathfrak{f}$. More precisely, to prove that $\mathfrak{L}_s \mathfrak{f}_*$ belongs to $\mathscr{I}_n$ if $\mathfrak{f}_*$ belongs to $\mathscr{I}_n$, let $\mathfrak{f}$ be given by (3.7) so that $\mathfrak{T}\mathfrak{f} = \mathfrak{f}_*$. By (3.10), $\mathfrak{T}\mathscr{L}_s \mathfrak{f} = \mathfrak{L}_s \mathfrak{f}_*$, which proves that $\mathfrak{L}_s \mathfrak{f}_*$ belongs to $\mathscr{I}$.

We turn now to an elementary identity to illustrate the fact that the space $\mathscr{I}$ enjoys some special properties. For every $\mathfrak{f}: \mathscr{E}_{*,1} \to \mathbb{R}$,

$$(\mathfrak{L}_- \mathfrak{f})(\phi) = -2 \sum_{x \neq 0} a(x)\ \mathfrak{f}(\{x\})$$

In particular, $(\mathfrak{L}_{-}\mathfrak{f})(\phi)=0$ for all $\mathfrak{f}$ in $\mathscr{I}_1$ because in this space $\mathfrak{f}(\{x\})=\mathfrak{f}(\{-x\})$ and $a(\cdot)$ is anti-symmetric. In contrast, $(\mathfrak{L}_{+}\mathfrak{g})(\{x\})=0$ for all functions $\mathfrak{g}\colon \mathscr{E}_{*,0}\to\mathbb{R}$ so that, for all $\mathfrak{f}$ in $\mathscr{I}_1$ and all $\mathfrak{g}\colon \mathscr{E}_{*,0}\to\mathbb{R}$,

$$\mathfrak{L}_{-}\mathfrak{f}=0, \qquad \mathfrak{L}_{+}\mathfrak{g}=0 \tag{7.2}$$

We turn now to the proof of some estimates involving the operators $\mathfrak{L}_s$, $\mathfrak{L}_d$, $\mathfrak{L}_-$, and $\mathfrak{L}_+$. The first lemma states that the operators $\mathfrak{L}_s$, $\mathfrak{L}_d$, $\mathfrak{L}_-$, and $\mathfrak{L}_+$ are bounded in $L^2(\mathscr{E}_{*,n})$ for each fixed $n\geqslant 1$.

Lemma 7.1. There exists a finite constant C_0 such that

$$\|\mathfrak{A}\mathfrak{f}\|_0^2\leqslant C_0 n^2\,\|\mathfrak{f}\|_0^2$$

for each $\mathfrak{f}$ in $L^2(\mathscr{E}_{*,n})$, were the operator $\mathfrak{A}$ stands for $\mathfrak{L}_s$, $\mathfrak{L}_d$, $\mathfrak{L}_+$, or $\mathfrak{L}_-$.

Proof. We only prove the estimate concerning $\mathfrak{L}_-$, the other ones being elementary. Fix a function $\mathfrak{f}\colon \mathscr{E}_{*,n}\to\mathbb{R}$ in L^2 and keep in mind that $\mathfrak{L}_-\mathfrak{f}$ maps $\mathscr{E}_{*,n-1}$ in $\mathbb{R}$. By the explicit expression for $\mathfrak{L}_-$ and a change of variables,

$$\|\mathfrak{L}_-\mathfrak{f}\|_0^2\leqslant 4\sum_{A\in\mathscr{E}_{*,n-1}}\Big\{\sum_{\substack{x,\,y\notin A\\ x,\,y\neq 0}} a(y-x)\,\mathfrak{f}(A\cup\{y\})\Big\}^2$$

Since $\sum_{x\in\mathbb{Z}^d} a(x)=0$, the previous expression is less than or equal to

$$8\sum_{A\in\mathscr{E}_{*,n-1}}\Big\{\sum_{\substack{y\notin A\\ y\neq 0}} a(y)\,\mathfrak{f}(A\cup\{y\})\Big\}^2+8\sum_{A\in\mathscr{E}_{*,n-1}}\Big\{\sum_{\substack{y\notin A,\,y\neq 0\\ x\in A}} a(y-x)\,\mathfrak{f}(A\cup\{y\})\Big\}^2$$

By Schwarz inequality, since $a(\cdot)$ is absolutely bounded and $\sum_{x\in\mathbb{Z}^d}|a(x)|$ is finite, this expression is less than or equal to

$$C_0 n\sum_{A\in\mathscr{E}_{*,n-1}}\sum_{\substack{y\notin A\\ y\neq 0}}\mathfrak{f}(A\cup\{y\})^2$$

for some finite constant C_0. To sum over A in $\mathscr{E}_{*,n-1}$ and over $y\neq 0$, $y\notin A$ is the same as to sum over B in $\mathscr{E}_{*,n}$ with a multiplicity factor n because all sets are counted n times. The previous expression is thus equal to

$$C_0 n^2\sum_{A\in\mathscr{E}_{*,n}}\mathfrak{f}(A)^2$$

which concludes the proof of the lemma. ∎

It follows from the next statement that the operators $\mathfrak{L}_d$ and $\mathfrak{L}_+ + \mathfrak{L}_-$ are anti-symmetric on $\mathscr{I}\cap L^2_{0,-1}(\mathscr{E}_*)$.

Lemma 7.2. For every $n \geqslant 1$ and every finitely supported functions $\mathfrak{u}, \mathfrak{v}: \mathscr{E}_{*,n} \to \mathbb{R}$

$$\langle \mathfrak{L}_d \mathfrak{u}, \mathfrak{v} \rangle = -\langle \mathfrak{u}, \mathfrak{L}_d \mathfrak{v} \rangle$$

For every finitely supported functions $\mathfrak{f}$, $\mathfrak{g}$ in $\mathscr{I}_{n-1}$, $\mathscr{I}_n$ respectively,

$$\frac{1}{n+1} \langle \mathfrak{L}_+ \mathfrak{f}, \mathfrak{g} \rangle = -\frac{1}{n} \langle \mathfrak{f}, \mathfrak{L}_- \mathfrak{g} \rangle.$$

Proof. The first identity is elementary and relies on the fact that $\sum_{x, y \in A} a(y-x) = 0$. Note, however, that both pieces of the operator are needed.

The proof of the second statement is more demanding. Fix finitely supported functions $\mathfrak{f}$, $\mathfrak{g}$ in $\mathscr{I}_{n-1}$, $\mathscr{I}_n$, respectively. By the explicit form of $\mathfrak{L}_+$,

$$\langle \mathfrak{g}, \mathfrak{L}_+ \mathfrak{f} \rangle = 2 \sum_{A \in \mathscr{E}_{*,n}} \sum_{x, y \in A} a(y-x)\, \mathfrak{g}(A)\, \mathfrak{f}(A \setminus \{y\})$$
$$+2 \sum_{A \in \mathscr{E}_{*,n}} \sum_{x \in A} a(x)\, \mathfrak{g}(A) \{ \mathfrak{f}(A \setminus \{x\}) - \mathfrak{f}(S_x[A \setminus \{x\}]) \}$$

Since $S_x[A \setminus \{x\}] = S_x A \setminus \{-x\}$ and since $\mathfrak{g}(S_x A) = \mathfrak{g}(A)$ for x in A because $\mathfrak{g}$ belongs to $\mathscr{I}_n$, a change of variables $B = S_x A$, $x' = -x$ in the second piece of the second term permits to rewrite the second term on the right hand side as

$$4 \sum_{x \neq 0} a(x) \sum_{\substack{A \in \mathscr{E}_{*,n} \\ A \ni x}} \mathfrak{g}(A)\, \mathfrak{f}(A \setminus \{x\})$$

because $a(-x) = -a(x)$. We claim that

$$\sum_{x \neq 0} a(x) \sum_{\substack{A \in \mathscr{E}_{*,n} \\ A \ni x}} \mathfrak{g}(A)\, \mathfrak{f}(A \setminus \{x\}) = \frac{1}{n-1} \sum_{x, y \neq 0} a(y-x) \sum_{\substack{A \in \mathscr{E}_{*,n} \\ A \ni x, y}} \mathfrak{g}(A)\, \mathfrak{f}(A \setminus \{y\}) \tag{7.3}$$

We conclude the proof of the lemma assuming (7.3), whose proof is presented at the end. It follows from identity (7.3) and the previous expression for $\langle \mathfrak{g}, \mathfrak{L}_+ \mathfrak{f} \rangle$ that

$$\langle \mathfrak{g}, \mathfrak{L}_+ \mathfrak{f} \rangle = 2 \left(1 + \frac{1}{n} \right) \sum_{A \in \mathscr{E}_{*,n}} \sum_{x, y \in A} a(y-x)\, \mathfrak{g}(A)\, \mathfrak{f}(A \setminus \{y\})$$
$$+2 \left(1 + \frac{1}{n} \right) \sum_{A \in \mathscr{E}_{*,n}} \sum_{x \in A} a(x)\, \mathfrak{g}(A)\, \mathfrak{f}(A \setminus \{x\})$$

The first term of the right hand side, which can be written as

$$2\left(1+\frac{1}{n}\right)\sum_{y\neq 0}\ \sum_{\substack{A\in\mathscr{E}_{*,n}\\ A\ni y}} \mathfrak{g}(A)\,\mathfrak{f}(A\backslash\{y\})\sum_{x\in A} a(y-x)$$

is equal to

$$-2\left(1+\frac{1}{n}\right)\sum_{y\neq 0} a(y)\sum_{\substack{A\in\mathscr{E}_{*,n}\\ A\ni y}} \mathfrak{g}(A)\,\mathfrak{f}(A\backslash\{y\})$$
$$-2\left(1+\frac{1}{n}\right)\sum_{x,\,y\neq 0} a(y-x)\sum_{\substack{A\in\mathscr{E}_{*,n}\\ A\ni y,\,A\not\ni x}} \mathfrak{g}(A)\,\mathfrak{f}(A\backslash\{y\})$$

because $\sum_{x\in A} a(y-x)=-a(y)-\sum_{x\neq 0,\,x\notin A} a(y-x)$. The first term of this formula cancels with the second one in the last expression for $\langle\mathfrak{g},\mathfrak{L}_+\mathfrak{f}\rangle$. Therefore,

$$\langle\mathfrak{g},\mathfrak{L}_+\mathfrak{f}\rangle=-2\left(1+\frac{1}{n}\right)\sum_{x,\,y\neq 0} a(y-x)\sum_{\substack{A\in\mathscr{E}_{*,n}\\ A\ni y,\,A\not\ni x}} \mathfrak{g}(A)\,\mathfrak{f}(A\backslash\{y\})$$

To conclude the proof of the lemma, it remains to change variables $B=A\backslash\{y\}$ and to recall the definition of the operator $\mathfrak{L}_-$.

We turn now to the proof of Claim (7.3). Since for y in A, $\mathfrak{g}(A)=\mathfrak{g}(S_yA)$ and since $|A|=n$, the left hand side of (7.3) is equal to

$$\frac{1}{n}\sum_{x\neq 0} a(x)\sum_{\substack{A\in\mathscr{E}_{*,n}\\ A\ni x}}\ \sum_{\substack{y\in A\cup\{0\}\\ y\neq x}} \mathfrak{g}(S_yA)\,\mathfrak{f}(A\backslash\{x\})$$
$$=\frac{1}{n}\sum_{\substack{x,\,y\neq 0\\ y\neq x}} a(x)\sum_{\substack{A\in\mathscr{E}_{*,n}\\ A\ni x,\,y}} \mathfrak{g}(S_yA)\,\mathfrak{f}(A\backslash\{x\})+\frac{1}{n}\sum_{x\neq 0} a(x)\sum_{\substack{A\in\mathscr{E}_{*,n}\\ A\ni x}} \mathfrak{g}(A)\,\mathfrak{f}(A\backslash\{x\})$$

Notice that the second term on the right hand side is precisely the original one. Consider the first term. Perform a change of variables $B=S_yA$, rewrite $(S_{-y}A)\backslash\{x\}$ as $S_{-y}(A\backslash\{x-y\})$ and recall that $\mathfrak{f}(S_{-y}(A\backslash\{x-y\})=\mathfrak{f}(A\backslash\{x-y\})$ if $-y$ belongs to A because $\mathfrak{f}$ is in $\mathscr{I}_{n-1}$, to rewrite this expression as

$$\frac{1}{n}\sum_{\substack{x,\,y\neq 0\\ y\neq x}} a(x)\sum_{\substack{A\in\mathscr{E}_{*,n}\\ A\ni x-y,\,-y}} \mathfrak{g}(A)\,\mathfrak{f}(A\backslash\{x-y\})$$

A change of variables $x' = x - y$, $y' = -y$, shows that this expression is equal to

$$\frac{1}{n} \sum_{x, y \neq 0} a(x-y) \sum_{\substack{A \in \mathscr{E}_{*,n} \\ A \ni x, y}} \mathfrak{g}(A)\, \mathfrak{f}(A \setminus \{x\})$$

To prove (7.3), it remains to recollect all previous identities. ∎

It follows from this result that the operator $\mathfrak{L}_+ + \mathfrak{L}_-$ is anti-symmetric in $L^2_{0,-1}(\mathscr{E}_*) \cap \mathscr{I}$:

Corollary 7.3. The operator $\mathfrak{L}_+ + \mathfrak{L}_-$ is anti-symmetric in $L^2_{0,-1}(\mathscr{E}_*) \cap \mathscr{I}$:

$$\langle\!\langle \mathfrak{f}, (\mathfrak{L}_+ + \mathfrak{L}_-)\, \mathfrak{g} \rangle\!\rangle_{0,-1} = -\langle\!\langle (\mathfrak{L}_+ + \mathfrak{L}_-)\, \mathfrak{f}, \mathfrak{g} \rangle\!\rangle_{0,-1}$$

for all finitely supported functions $\mathfrak{f}$, $\mathfrak{g}$ in $\mathscr{I}$. The same statement remains in force if $\mathfrak{L}_+ + \mathfrak{L}_-$ is replaced by $\Pi_n(\mathfrak{L}_+ + \mathfrak{L}_-)\, \Pi_n$ for every $n \geqslant 1$, where $\Pi_n = \sum_{1 \leqslant j \leqslant n} \pi_j$.

The proof of Corollary 7.3 is elementary and left to the reader. One needs only to recall identities (7.2). The next result states that $\mathfrak{L}_d$ is a bounded operator from $L^2(\mathscr{E}_{*,n})$ to $\mathfrak{H}_{-1}(\mathscr{E}_{*,n})$.

Lemma 7.4. There exists a finite constant C_0, independent of n, such that for each $n \geqslant 1$ and for any finitely supported functions $\mathfrak{f}$, $\mathfrak{g}$: $\mathscr{E}_{*,n} \to \mathbb{R}$,

$$\langle \mathfrak{L}_d \mathfrak{f}, \mathfrak{g} \rangle \leqslant C_0 \sqrt{n}\, \|\mathfrak{g}\|_1\, \|\mathfrak{f}\|_0$$

In particular,

$$\|\mathfrak{L}_d \mathfrak{f}\|_{-1} \leqslant C_0 \sqrt{n}\, \|\mathfrak{f}\|_0$$

The very same estimates remain in force if $\mathfrak{L}_d$ is replaced by $\mathfrak{L}_s$.

The proof of Lemma 7.4 is elementary and left to the reader. Next estimate is Lemma 2.3 of ref. 8. Recall from Section 4 the definition of the operator $\mathfrak{B}$.

Lemma 7.5. There exists a finite constant C_0 such that for any function $\mathfrak{f}$: $\mathscr{E}_{n,*} \to \mathbb{R}$ in $\mathfrak{H}_1$,

$$\|\mathfrak{f}\|_1^2 \leqslant \|\mathfrak{B}\mathfrak{f}\|^2_{\mathscr{X}_n, 1} \leqslant C_0 n\, \|\mathfrak{f}\|_1^2$$

Proof. The first inequality is elementary and follows from the explicit formulas for the respective H_1 norms. To prove the second inequality, for $\mathbf{x}=(x_1,..., x_n)\in \mathscr{E}_{n,*}$, let

$$W_1(\mathbf{x})=\sum_{j=1}^{n} s(x_j), \qquad W_2(\mathbf{x})=\sum_{i,j=1}^{n} s(x_i-x_j)$$

We also denote these quantities by $W_1(A)$ and $W_2(A)$, for $A=\{x_1,..., x_n\}$. A simple computation shows that there exists a finite constant C_0 such that

$$|\mathfrak{B}\mathfrak{L}_s\mathfrak{f}(\mathbf{x})-\tilde{\mathfrak{L}}_s\mathfrak{B}\mathfrak{f}(\mathbf{x})|\leqslant C_0\{W_1(\mathbf{x})+W_2(\mathbf{x})\}\ |\mathfrak{B}\mathfrak{f}(\mathbf{x})| \tag{7.4}$$

for every $\mathbf{x}$ in $\mathscr{E}_{n,*}$ and $\mathfrak{f}\colon \mathscr{E}_{n,*}\to\mathbb{R}$.

We are now in a position to prove the second bound. By definition,

$$\|\mathfrak{B}\mathfrak{f}\|^2_{\mathscr{X}_n,1}=-\frac{1}{n!}\sum_{\mathbf{x}\in\mathscr{X}_n}(\mathfrak{B}\mathfrak{f})(\mathbf{x})(\tilde{\mathfrak{L}}_s\mathfrak{B}\mathfrak{f})(\mathbf{x})$$

Since $\mathfrak{B}\mathfrak{f}$ vanishes outside $\mathscr{E}_{n,*}$, we may restrict the sum to $\mathscr{E}_{n,*}$. Now, adding and subtracting $(\mathfrak{B}\mathfrak{L}_s\mathfrak{f})(\mathbf{x})$ in this expression and recalling (7.4), we obtain that

$$\begin{aligned}\|\mathfrak{B}\mathfrak{f}\|^2_{\mathscr{X}_n,1}&\leqslant \|\mathfrak{f}\|_1^2+\frac{C_0}{n!}\sum_{\mathbf{x}\in\mathscr{E}_{n,*}}\{W_1(x)+W_2(x)\}\{\mathfrak{B}\mathfrak{f}(\mathbf{x})\}^2\\&=\|\mathfrak{f}\|_1^2+C_0\sum_{A\in\mathscr{E}_{n,*}}\{W_1(A)+W_2(A)\}\ \mathfrak{f}(A)^2\end{aligned}$$

By Theorem 4.7 in ref. 8 or Lemma 3.7 in ref. 13 the second term of the previous formula is bounded by $C_0 n\,\|\mathfrak{f}\|_1^2$, which concludes the proof of the lemma. ▮

It follows from this result that

$$\frac{1}{C_0 n}\|\mathfrak{f}\|^2_{-1}\leqslant \|\mathfrak{B}\mathfrak{f}\|^2_{\mathscr{X}_n,-1}\leqslant\|\mathfrak{f}\|^2_{-1} \tag{7.5}$$

Lemma 7.6. There exists a finite constant C_0 such that

$$\|\mathfrak{B}\mathfrak{L}_s\mathfrak{f}-\tilde{\mathfrak{L}}_s\mathfrak{B}\mathfrak{f}\|^2_{\mathscr{X}_n,-1}\leqslant C_0 n^2\,\|\mathfrak{f}\|_1^2, \qquad \|\mathfrak{B}\mathfrak{L}_d\mathfrak{f}-\tilde{\mathfrak{L}}_d\mathfrak{B}\mathfrak{f}\|^2_{\mathscr{X}_n,-1}\leqslant C_0 n^2\,\|\mathfrak{f}\|_1^2$$

for all $n\geqslant 1$ and all functions $\mathfrak{f}\colon\mathscr{E}_{n,*}\to\mathbb{R}$.

Proof. We prove the first estimate and leave to the reader the details of the second. Fix $n \geqslant 1$ and a function $\mathfrak{h}\colon \mathcal{X}_n \to \mathbb{R}$. We need to estimate the scalar product

$$\frac{1}{n!} \sum_{\mathbf{x} \in \mathcal{X}_n} \mathfrak{h}(\mathbf{x}) \{ \mathfrak{B} \mathfrak{L}_s \mathfrak{f}(\mathbf{x}) - \tilde{\mathfrak{L}}_s \mathfrak{B} \mathfrak{f}(\mathbf{x}) \} \tag{7.6}$$

in terms of the $H_1(\mathcal{X}_n)$ norm of $\mathfrak{h}$ and the $H_1(\mathcal{E}_{n,*})$ norm of $\mathfrak{f}$. There are two possible cases. Either $\mathbf{x}$ belongs to $\mathcal{E}_{n,*}$ or $\mathbf{x}$ does not belong to $\mathcal{E}_{n,*}$.

In the first case, by (7.4), the expression inside braces in previous formula is absolutely bounded by $C_0 W_*(\mathbf{x}) \, |\mathfrak{B}\mathfrak{f}(\mathbf{x})|$ for some finite constant C_0, where $W_*(\mathbf{x}) = W_1(\mathbf{x}) + W_2(\mathbf{x})$. Therefore, the corresponding piece in the previous formula is bounded above by

$$\begin{aligned} &\frac{1}{n!} \sum_{\mathbf{x} \in \mathcal{E}_{n,*}} W_*(\mathbf{x}) \, |\mathfrak{h}(\mathbf{x})| \, |\mathfrak{B}\mathfrak{f}(\mathbf{x})| \\ &\qquad \leqslant \frac{1}{2\ell} \frac{1}{n!} \sum_{\mathbf{x} \in \mathcal{E}_{n,*}} W_*(\mathbf{x}) \, \mathfrak{h}(\mathbf{x})^2 + \frac{\ell}{2} \sum_{A \in \mathcal{E}_{n,*}} W_*(A) \, \mathfrak{f}(A)^2 \end{aligned}$$

for every $\ell > 0$.

If $\mathbf{x}$ does not belong to $\mathcal{E}_{n,*}$, the corresponding piece of the scalar product writes

$$-\frac{2}{n!} \sum_{\substack{\mathbf{x} \notin \mathcal{X}_n \\ z \in \mathbb{Z}^d \\ 1 \leqslant j \leqslant n}} s(z) \, \mathfrak{h}(\mathbf{x}) \, \mathfrak{B}\mathfrak{f}(\mathbf{x} + z\mathbf{e}_j) - \frac{2}{n!} \sum_{\substack{\mathbf{x} \notin \mathcal{X}_n \\ z \in \mathbb{Z}^d}} s(z) \, \mathfrak{h}(\mathbf{x}) \, \mathfrak{B}\mathfrak{f}(\mathbf{x} + z\mathbf{1})$$

because in this case $\mathfrak{B}\mathfrak{L}_s \mathfrak{f}(\mathbf{x}) = 0$.

We estimate the first term and claim that the second can be handled in the same way. Since $\mathfrak{B}\mathfrak{f}$ vanishes outside $\mathcal{E}_{n,*}$, it is implicit in the previous formula that the first sum is restricted to all $\mathbf{x}$ such that $\mathbf{x} + z\mathbf{e}_j$ belongs to $\mathcal{E}_{n,*}$. Since $\mathbf{x} + z\mathbf{e}_j \in \mathcal{E}_{n,*}$ and $\mathbf{x} \notin \mathcal{E}_{n,*}$, either $x_j = 0$ or $x_j = x_k$ for some k. In particular, since $2ab \leqslant \ell a^2 + \ell^{-1} b^2$ for every $\ell > 0$, a change of variables gives that the first term of the previous formula is bounded above by

$$\frac{1}{n! \ell} \sum_{\mathbf{x} \in \mathcal{X}_n} \mathfrak{h}(\mathbf{x})^2 \, \tilde{W}_*(\mathbf{x}) + \frac{\ell}{n!} \sum_{\mathbf{x} \in \mathcal{E}_{n,*}} \mathfrak{B}\mathfrak{f}(\mathbf{x})^2 \, W_*(\mathbf{x})$$

where $\tilde{W}_*(\mathbf{x}) = \sum_{1 \leqslant j \leqslant n} \mathbf{1}\{x_j = 0\} + \sum_{j \neq k} \mathbf{1}\{x_j = x_k\}$. We may of course replace the sum over $\mathcal{X}_n$ by a sum over $\mathcal{E}_{n,*}$ in the second term, loosing the factor $n!$.

Adding together all previous estimates, we obtain that the scalar product (7.6) is bounded above by

$$\frac{C_0}{n!\ell} \sum_{\mathbf{x} \in \mathscr{X}_n} \mathfrak{h}(\mathbf{x})^2 \{\tilde{W}_*(\mathbf{x}) + W_*(\mathbf{x})\} + C_0 \ell \sum_{A \in \mathscr{E}_{n,*}} \mathfrak{f}(A)^2 W_*(A).$$

The same proof of Theorem 4.7 in ref. 8 or Lemma 3.7 in ref. 13 shows that in dimension $d \geqslant 3$, the first term is bounded above by $C_1 n A^{-1} \|\mathfrak{h}\|^2_{\mathscr{X}_n, 1}$ for some finite constant C_1, while the second, by the quoted results, is less than or equal to $C_1 n A \|\mathfrak{f}\|_1^2$. To conclude the proof of the lemma, it remains to minimize over A and to recall the variational formula for the H_{-1} norm. ▮

We conclude this section with a central estimate involving the operators $\mathfrak{L}_+$, $\mathfrak{L}_-$. This result is Theorem 4.4 in ref. 8. The reader can find in Lemma 4.1 of ref. 13 a clearer proof.

Lemma 7.7. There exists a finite constant C_0 depending only on the transition probability p such that

$$\Big\{ \sum_{A \in \mathscr{E}_{*, n+1}} (\mathfrak{L}_+ \mathfrak{f})(A)\, \mathfrak{g}(A) \Big\}^2 \leqslant C_0 n \|\mathfrak{f}\|_1^2 \|\mathfrak{g}\|_1^2$$

$$\Big\{ \sum_{A \in \mathscr{E}_{*, n}} \mathfrak{f}(A) (\mathfrak{L}_- \mathfrak{g})(A) \Big\}^2 \leqslant C_0 n \|\mathfrak{f}\|_1^2 \|\mathfrak{g}\|_1^2$$

for all $n \geqslant 1$ and all finite supported functions $\mathfrak{f}\colon \mathscr{E}_{*,n} \to \mathbb{R}$, $\mathfrak{g}\colon \mathscr{E}_{*,n+1} \to \mathbb{R}$. In particular,

$$\|\mathfrak{L}_+ \mathfrak{f}\|_{-1}^2 \leqslant C_0 n \|\mathfrak{f}\|_1^2, \qquad \|\mathfrak{L}_- \mathfrak{g}\|_{-1}^2 \leqslant C_0 n \|\mathfrak{g}\|_1^2$$

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